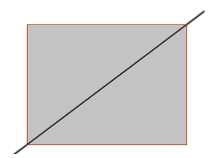
FAU Math Circle
1/17/2015
SOLUTIONS

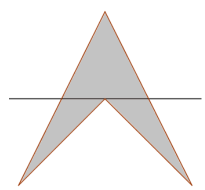
Math Warm Up

1. Jeff rides his bike home from the library at 6 miles per hour. At the same time his brother Mike walks from their home to the library at 3.5 miles per hour. The home and the library are 2 miles apart. Which brother will be closer to home when they run into each other? **Solution.** Both are at the same distance!

2. A quadrilateral is a polygon with four sides. The quadrilateral on the right (a rectangle) is divided by a straight line into two triangles. Can you think of a quadrilateral that can be divided by a straight line into three triangles?



Solution.



3. Moving a single digit, make this equality correct:

$$101 - 102 = 1$$

A hint to be given is that digits can also be moved vertically.

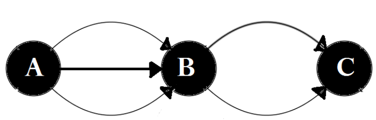
Solution. $101 - 10^2 = 1$.

Today's Problems

We will try something a little bit different today. We'll do several problems in order since each problem will build up a little on the previous one, then (if there is time) we'll do problems the usual way. They all have to do with counting. Most of these problems have been copied (sometimes slightly altered) from *Mathematical Circle Diaries, Year 1,* by Anna Burago.

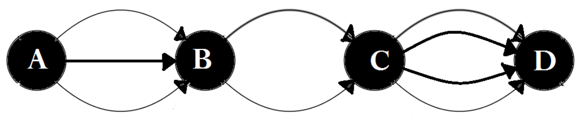
Part I

1. The towns of Alpha (A), Beta (B), and Cornucopia (C) in the south of the country of Nosuchplacia are connected by one way roads as shown. Why they are only one way, we don't know, but one can only travel them in the direction of the arrows, from left to right. How many different routes can one take from Alpha to Cornucopia?



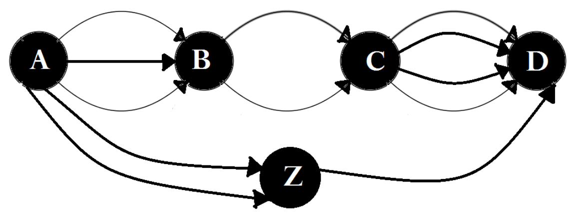
Solution. There are 3 routes from A to B, for each one of these choices there are two from B to C. $\boxed{\text{The answer is } 3 \times 2 = 6.}$

2. A new town, Delta (D) is connected to the other towns by some roads, as shown. Again all traffic is in only one direction. In how many ways can one travel from Alpha to Delta?



Solution. For each way we can travel from A to C, there are 4 ways to get to D. $\boxed{\text{The answer is } 3 \times 2 \times 4 = 24}$

3. An even newer town, Zamboni (Z) is connected by routes to the previous towns, as shown. In how many different ways can one travel from Alpha to Delta now?

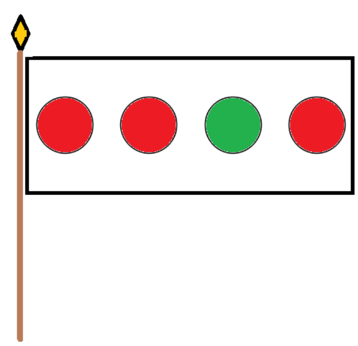


Solution. The routes can be divided into two separate sets: Not through Z and through Z. Not through Z is the same as before, $3 \times 2 \times 4 = 12$ ways. Through Z, there are $2 \times 1 = 2$ ways. $\boxed{\text{The answer is } 24 + 2 = 26.}$

4. Alice has 5 pairs of shoes, 3 pairs of skirts and 7 pairs of blouses. How many different outfits can Alice put together?

Solution. For each pair of shoes Alice can put on of 3 skirts, and for every choice of shoes and skirt, she has 7 blouses. $\boxed{\text{The answer is } 5 \times 3 \times 7 = 105.}$

5. Ella's hopscotch team has designed a flag consisting of 4 circles on a white background. The circles are to be painted either red or green. How many different patterns can be formed. For example, one pattern would be RRRR. The picture shows another possible pattern, RRGR.



Solution. We have two choices for the first circle, for each such choice we have two choices for the second circle, then 2 choices for the third circle, ... $\boxed{\text{The answer is } 2 \times 2 \times 2 \times 2 = 16.}$

How many different patterns if the flag is to have 5 circles? 6 circles?

How many different patterns if three colors (red, green, and blue) are allowed and the flag is to have 7 circles?

Solution. $\boxed{\text{The answer is } 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^7 = 128.}$

1 Part II

1. License plate numbers in Nosuchplacia consist of 3 digits, followed by two letters, followed by two **different** *odd* digits. Here are some examples of legal license plates.

123AD97	000GG15	103BZ73	888EE51

   But the two plates below are illegal.

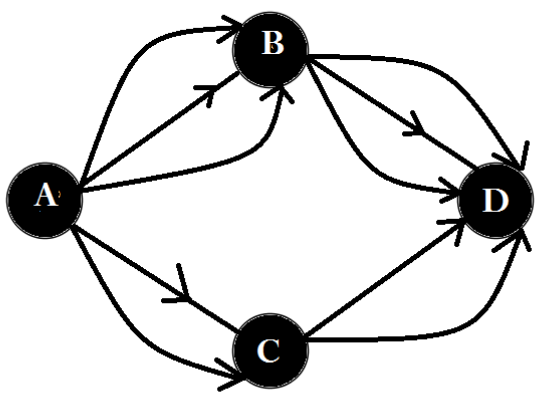
123AB97	000CD55	022AZ17	454WW31

   How many possible different license plates can there be in Nosuchplacia?

   Solution. A difficulty here are the last two digits and maybe one should answer this first, in how many ways can we write out two distinct odd digits? Of course, we could write them all out beginning with 13 and ending with 97 and count them. But here is a better way. There are 5 odd digits: 1, 3, 5, 7, 9. You can choose the first one arbitrarily, so there are five choices for the first digit. However we choose the first one, that leaves four choices for the second digit. So there is a total of $5 \times 4 = 20$ choices.

   $\boxed{\text{The answer is } 10 \times 10 \times 10 \times 26 \times 26 \times 20 = 13520000.}$

2. Four towns, A, B, C, D, are connected by routes as shown. In how many different ways can one travel from A to D? Only travel in the direction of the arrows is allowed.

   Solution. We can go from A to D over C or over B. Over C there are $3 \times 3 = 9$ ways. Over C, $2 \times 2 = 4$ ways. $\boxed{\text{The answer is } 9 + 4 = 13.}$

3. Prince Ivan is on a quest to free Princess Masha, who has been imprisoned in the castle. The castle door has a simple digital lock with ten buttons, numbered 0 to 9. The door is guarded by a hungry dragon, Pashka, who likes hot dogs.The door can be opened by typing a secret 4 digit code, and Pashka can be distracted by hot dogs. It takes Ivan 1 second to try out a single 4-digit combination, and it takes 20 seconds for Pashka to gulp down a single hot dog. After Ivan opens the lock it will take him one minute to get Masha and fly off with her on his magic flying carpet

(a) How many hot dogs should Ivan packfor the quest if he wants to be sure to fly out of the castle alive with Masha? He should have enough hot dogs just incase he is really unlucky and he has to go through all possible combinations before hitting on the right code.
(b) Suppose that Ivan knows in advance that the secret 4-digit code is composed only of odd digits. How many hot dog swill he need now?
(c) Suppose that Ivan knows in advance that the secret 4-digit code is composed of odd digits only and has exactly one digit 5 in it. How many hot dogs will he need in this case?

Solution. **(a)** There will be $10 \times 10 \times 10 \times 10 = 10,000$ different combinations, so Ivan will need to be prepared for 10,000 seconds plus the one minute to take Masha out. Thus he will need $\boxed{10060/20 = 503 \text{ hot dogs.}}$

(b) There being only five odd integers the total number of combinations is reduced to $5 \times 5 \times 5 \times 5 = 625$. Adding the 60 seconds, Masha and Ivan will be out in 685 seconds, so he will need $\boxed{685/20 = 34.25 \text{ hot dogs,}}$ 34 hot dogs and a quarter of a hot dog. Better to carry 35.

(c) The 5 can be in either first, second, third, or fourth place. In any case that leaves a choice of four odd digits for the remaining three places. Wherever 5 is, there are $4 \times 4 \times 4 = 64$ combinations with 5 in that place. That gives a total of $64+64+64+64 = 256$ combinations, so 256 seconds (in the worst case scenario) to open the lock. Adding the extra minute needed we get 316 seconds, thus Ivan should carry $\boxed{316/20 = 15.8 \text{ hot dogs.}}$

4. Al's pizzeria and pet grooming salon offers the following toppings for its delicious boiled pizza: onions, garlic, anchovies, chocolate. You can have a pizza with any combination of these toppings, or with no topping at all. In how many different ways can you have your pizza? The order of the toppings does not matter. Sez Al: We only use the purest spring water to boil our pizzas.

   Solution. We can make a list here of possible toppings

   (a) No topping–1 way
   (b) One topping– 4 ways
   (c) Two toppings: {O,G}, {O,A}, {O,C}, {G, C}, {G,A}, {A,C}–6 ways
   (d) Three toppings: {O,G, A}, {O,G, C}, {O, A, C}, {G,A, C}–4 ways
   (e) Four toppings: 1 way.

   $\boxed{\text{The answer is } 1 + 4 + 6 + 4 + 1 = 16.}$

5. (a) There are 3 books on a shelf. In how many ways can the books be arranged in a different order so that no one book remains in its original place? (b) What if there are 4 books on the shelf? (c) What if there are 5?

   Solution. (a) Label the books 1, 2, 3. Book 1 can go to position of 2 or of 3. If to position 2, 2 has to go to 3, 3 to 1. If to position 3, 3 has to go to 2 and 2 to 1. Thus $\boxed{\text{two ways in all.}}$ (b) Book 1 can go to positions 2, 3or 4. The number of ways of what happens next is the same, so we can assume 1 goes to 3, and multiply the end result by 3. Assume thus 1 goes to 4. Place 4 temporarily in position 1 so the books are now in order 4231. Any rearrangement of 432 that leaves no book in the current position gives a rearrangement of all 4 books leaving none in the very first original position. From part (a), there are 2 such rearrangements. But 4321 also has all books rearranged so no book is in its original position. All in all 3 ways, so for the original question $\boxed{\text{9ways}}$. (C) A similar reasoning shows there 44 such arrangements. In general, if you have n books, the first book can go to any of positions 2, 3, ,4, etc. The number of ways you can then arrange the remaining books is always the same, so we may as well assume we send the first book to position n, see how many rearrangements we get that way, and then multiply by $n-1$. So place the first book in position n, the book in n gets placed temporarily in position 1. Any rearrangement that leaves no books fixed of books $n, 2, \ldots, n-1$ will be a sought for rearrangement. These give us all the rearrangements that have book 1 in position n, book n in all positions EXCEPT position 1. If we add to these all possible rearrangements of books $2, \ldots, n-1$ leaving none fixed, we have all the rearrangements with book 1 going to place n. The end result is that the number of rearrangements of n books ($n \geq 3$) leaving no book fixed is $(n-1) \times (a+b)$ where a is the number of such rearrangements of $n-1$ books, b of $n-2$ books. In particular, if $n = 5$, having seen that the number for 4 books was 9 and for 3 books was 2, for 5 books it will be $4 \times (9+2) = 44$.

6. Here is something completely different. One hundred numbers are placed around a circle in such a way that every number is the arithmetic mean of its two neighbors. Prove that all the numbers must be equal.

   Solution. Suppose they are not all equal. Then one of these numbers will be the largest and, because not all numbers are equal there is at least one number $<$ than this largest number. Starting at a number less than the largest number go around the circle (clockwise for example) until you hit the largest number for the first time. So you now have the largest number and at least one of its neighbors is less than it. But this cannot be, this would make the largest number the arithmetic mean of two numbers, one less than it, the other one less than or equal it.