STA 6446 Applied Time Series Assignment 3

2-5. By using Cauchy-Schwarz inequality, we have

$$E\sum_{j=1}^{\infty}|\theta|^j|X_{n-j}| \le \sum_{j=1}^{\infty}|\theta|^j E|X_{n-j}|$$

$$\le \sum_{j=1}^{\infty}|\theta|^j\sqrt{E[X_{n-j}^2]}$$

$$= \sum_{j=1}^{\infty}|\theta|^j\sqrt{\gamma(0)+\mu^2} < \infty \quad \text{since } |\theta|<1.$$

So, $\sum_{j=1}^{\infty}|\theta|^j|X_{n-j}|<\infty$ with probability 1.

Next, we prove that $\sum_{j=1}^{m}\theta^j X_{n-j}$ converges in mean squares as $m\to\infty$.

Note that $m>k$,

$$E|S_m - S_k|^2 = E\left(\sum_{r=k+1}^{m}\theta^r X_{n-r}\right)^2$$

$$= \sum_{s=k+1}^{m}\sum_{r=k+1}^{m}\theta^{r+s}E(X_{n-r}X_{n-s})$$

$$= \sum_{s}\sum_{r}\theta^{r+s}(\gamma(r-s)+\mu^2)$$

$$\le \sum_{s}\sum_{r}|\theta|^{r+s}(\gamma(0)+\mu^2)$$

$$=(\gamma(0)+\mu^2)\left(\sum_{r=k+1}^{m}|\theta|^r\right)^2$$

$$\to 0 \quad \text{as } k,m\to\infty \quad \text{since } \sum_{r=0}^{\infty}|\theta|^r<\infty.$$

So, S_m converges in mean squares as $m\to\infty$.

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2.5. By using Cauchy–Schwarz inequality, we have

$$E \sum_{j=1}^{\infty} |\theta|^j |X_{n-j}| \le \sum_{j=1}^{\infty} |\theta|^j E|X_{n-j}|$$

$$\le \sum_{j=1}^{\infty} |\theta|^j \sqrt{E[X_{n-j}^2]}$$

$$= \sum_{j=1}^{\infty} |\theta|^j \sqrt{\gamma(0) + \mu^2} < \infty \quad \text{since } |\theta|<1.$$

So, $\sum_{j=1}^{\infty} |\theta|^j |X_{n-j}| < \infty$ with probability 1.

Next, we prove that $\sum_{j=1}^{m} \theta^j X_{n-j}$ converges in mean squares as $m \to \infty$.

Note that $m > k$,

$$E|S_m - S_k|^2 = E\left(\sum_{r=k+1}^{m} \theta^r X_{n-r}\right)^2$$

$$= \sum_{s=k+1}^{m} \sum_{r=k+1}^{m} \theta^{r+s} E(X_{n-r} X_{n-s})$$

$$= \sum_s \sum_r \theta^{r+s} (\gamma(r-s) + \mu^2)$$

$$\le \sum_s \sum_r |\theta|^{r+s} (\gamma(0) + \mu^2)$$

$$= (\gamma(0)+\mu^2) \left(\sum_{r=k+1}^{m} |\theta|^r\right)^2$$

$$\to 0 \quad \text{as } k, m \to \infty \quad \text{since } \sum_{r=0}^{\infty} |\theta|^r < \infty.$$

So, S_m converges in mean squares as $m \to \infty$.

2.8. By recursive calculation, we find

$$X_t = \phi X_{t-1} + Z_t$$
$$= Z_t + \phi(Z_{t-1} + \phi X_{t-2})$$
$$= \cdots$$
$$= Z_t + \phi Z_{t-1} + \cdots + \phi^n Z_{t-n} + \phi^{n+1} X_{t-n-1}.$$

$$\Leftrightarrow \quad X_t - \phi^{n+1} X_{t-n-1} = Z_t + \phi Z_{t-1} + \cdots + \phi^n Z_{t-n}.$$

We calculate the variances of both sides.

$$Var(X_t - \phi^{n+1} X_{t-n-1}) = Cov(X_t - \phi^{n+1} X_{t-n-1}, X_t - \phi^{n+1} X_{t-n-1})$$

$$= Cov(X_t, X_t) - 2\phi^{n+1} Cov(X_t, X_{t-n-1}) + \phi^{2n+2} Cov(X_{t-n-1}, X_{t-n-1})$$

$$= \gamma(0)(1 + \phi^{2n+2}) - 2\phi^{n+1}\gamma(n+1)$$

$$\le \gamma(0)(1 + 2|\phi|^{n+1} + |\phi|^{2n+2})$$

$$= 4\gamma(0) \qquad \text{if } \{X_t\} \text{ is stationary and } |\phi| = 1.$$

$Var(Z_t + \phi Z_{t-1} + \cdots + \phi^n Z_{t-n}) = n\sigma^2$ if $|\phi| = 1$.

obviously, $n\sigma^2$ is not less than or equal to $4\gamma(0)$ for all n. Thus, $\{X_t\}$ can not be stationary if $|\phi| = 1$.

2.10. $X_t - \phi X_{t-1} = Z_t + \theta Z_{t-1}$ with $\phi = 0.5$, $\theta = 0.5$.

By equation (2.3.3) in section 2.3, we find

$$X_t = \sum_{j\geq 0}^{\infty} \psi_j Z_{t-j},$$

where $\psi_0 = 1$, $\psi_j = (\phi+\theta)\phi^{j-1}$, $j \geq 1$

$$= (0.5)^{j-1}, \quad j \geq 1$$

By equation (2.3.5) in section 2.3, we have

$$Z_t = \sum_{j\geq 0}^{\infty} \pi_j X_{t-j},$$

where $\pi_0 = 1$, $\pi_j = -(\phi+\theta)(-\theta)^{j-1} = -(-0.5)^{j-1}$, $j \geq 1$.

Agrees with ITSM.

2.12. MA(1) : $X_t = Z_t - 0.6 Z_{t-1}$, $\{Z_t\} \sim WN(0,1)$.

$$\bar{x}_{100} = 0.157.$$

A 95% confidence interval for μ is $\bar{X}_{100} \pm 1.96\sqrt{\operatorname{Var}\bar{X}_{100}}$.

Note that $\operatorname{Var}\bar{X}_{100} = \frac{1}{n}\sum_{-n}^{n}\left(1 - \frac{|h|}{n}\right)\gamma(h)$

$$= \frac{1}{100}\left[\gamma(0) + 2 \times \frac{99}{100}\gamma(1)\right]$$

$$= \frac{1}{100}\left[1.36 - 1.98 \times 0.6\right] = 0.00172$$

So, 95% confidence bounds for μ are approximately

$$\bar{x}_{100} \pm 1.96\sqrt{0.00172} = 0.157 \pm 1.96\times(0.0415)$$
$$= 0.157 \pm 0.0813 \quad \text{or} \quad [0.076,\ 0.238].$$

Since the 95% confidence interval for μ does not include zero, we should reject $H_0: \mu = 0$.

2.13. (a) $\hat{\rho}(1) = 0.438$, $\hat{\rho}(2) = 0.145$.

AR(1): $X_t - \phi X_{t-1} = Z_t$

By the Bartlett formula, $\operatorname{Var}\hat{\rho}(1) \simeq \frac{1}{n}(1-\phi^2)$

$$\operatorname{Var}\hat{\rho}(2) \simeq \frac{1}{n}\left[(1+\phi^2)^2 - 4\phi^4\right].$$

So, the 95% approximate confidence intervals for

$\rho(1)$ is $\hat{\rho}(1) \pm \frac{1.96}{\sqrt{n}}(1-\phi^2)^{\frac{1}{2}}$

and c.I. for $\rho(2)$ is $\hat{\rho}(2) \pm \frac{1.96}{\sqrt{n}}(1-\phi^2)^{\frac{1}{2}}(1+3\phi^2)^{\frac{1}{2}}$.

Replace ϕ by $\hat{\phi} = \hat{\rho}(1) = 0.438$ and $\hat{\rho}(2) = 0.148$, we find

C.I. for $\rho(1)$: $0.438 \pm \frac{1.96}{\sqrt{100}} \times (1-0.438^2)^{\frac{1}{2}} = [0.262,\ 0.614]$

and

C.I. for $\rho(2)$: $0.148 \pm \frac{1.96}{\sqrt{100}} \times (1-0.438^2)^{\frac{1}{2}}(1+3\times 0.438^2)^{\frac{1}{2}} = [-0.073,\ 0.369]$.

We conclude that the data are not consistent with $\phi = 0.8$ since both $\rho(1) = 0.8$ and $\rho(2) = 0.64$ are outside the bounds.

(b) MA(1): $X_t = Z_t + \theta Z_{t-1}$.

By Bartlett formula, $\text{Var}\,\hat{\rho}(1) \simeq \frac{1}{n}(1-3\rho^2(1)+4\rho^4(1))$

$\text{Var}\,\hat{\rho}(2) \simeq \frac{1}{n}(1+2\rho^2(1))$.

So, the approximate 95% C.I. bounds:

$\rho(1)$: $\hat{\rho}(1) \pm \frac{1.96}{\sqrt{n}}(1-3\rho^2(1)+4\rho^4(1))^{1/2}$

$\rho(2)$: $\hat{\rho}(2) \pm \frac{1.96}{\sqrt{n}}(1+2\rho^2(1))^{1/2}$.

Setting $\rho(1)=\hat{\rho}(1)$, $n=100$, $\hat{\rho}(1)=0.438$, $\hat{\rho}(2)=0.148$, we find

$\rho(1)$: $0.438 \pm 0.196 \times 0.756 = [0.290, 0.586]$

$\rho(2)$: $0.148 \pm 0.196 \times 0.176 = [-0.082, 0.378]$.

For $\theta=0.6$, we have $\rho(1)=\frac{\theta}{1+\theta^2}=0.4412$, $\rho(2)=0$.

The bounds are consistent with $\rho(1)=0.4412$, $\rho(2)=0$ and hence the data are consistent with $X_t = Z_t + 0.6 Z_{t-1}$.

2.14. $X_t = A\cos(\omega t) + B\sin(\omega t)$, A, B are uncorrelated with mean 0 and variance 1.

(a) $P_1 X_2 = \phi_{11} X_1$

where $\gamma(0)\phi_{11} = \gamma(1) \Rightarrow \phi_{11} = \rho(1) = \cos\omega$

and $E(X_2 - P_1X_2)^2 = \gamma(0) - \phi_{11}\gamma(1) = \gamma(0)(1-\phi_{11}^2) = \gamma(0)(1-\cos^2\omega) = \sin^2\omega$.

(b) $$P_2 X_3 = \phi_{21} X_2 + \phi_{22} X_1$$

where $$\begin{bmatrix} \gamma(0) & \gamma(1) \\ \gamma(1) & \gamma(0) \end{bmatrix} \begin{bmatrix} \phi_{21} \\ \phi_{22} \end{bmatrix} = \begin{bmatrix} \gamma(1) \\ \gamma(2) \end{bmatrix},$$

i.e. $$\begin{bmatrix} 1 & \cos w \\ \cos w & 1 \end{bmatrix} \begin{bmatrix} \phi_{21} \\ \phi_{22} \end{bmatrix} = \begin{bmatrix} \cos w \\ \cos 2w \end{bmatrix}$$

So, $$\begin{bmatrix} \phi_{21} \\ \phi_{22} \end{bmatrix} = \begin{bmatrix} 1 & \cos w \\ \cos w & 1 \end{bmatrix}^{-1} \begin{bmatrix} \cos w \\ \cos 2w \end{bmatrix} = \frac{1}{1-\cos^2 w} \begin{bmatrix} 1 & -\cos w \\ -\cos w & 1 \end{bmatrix} \begin{bmatrix} \cos w \\ \cos 2w \end{bmatrix}$$

$$= \frac{1}{\sin^2 w} \begin{bmatrix} \cos w - \cos w \cos 2w \\ \cos 2w - \cos^2 w \end{bmatrix} = \frac{1}{\sin^2 w} \begin{bmatrix} \cos w (1 - \cos^2 w + \sin^2 w) \\ \cos^2 w - \sin^2 w - \cos^2 w \end{bmatrix}$$

$$= \begin{bmatrix} 2\cos w \\ -1 \end{bmatrix}.$$

Thus, $$P_2 X_3 = 2\cos w \, X_2 - X_1.$$

and $$E(X_3 - P_2 X_3)^2 = \gamma(0) - [\phi_{21} \ \phi_{22}] \begin{bmatrix} \gamma(1) \\ \gamma(2) \end{bmatrix}$$

$$= 1 - [2\cos w \ \ -1] \begin{bmatrix} \cos w \\ \cos 2w \end{bmatrix}$$

$$= 1 - 2\cos^2 w + \cos 2w = 0$$