The Inverse Eigenvalue Problem of a Graph, An Overview

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Let $G = (V, E)$ be an undirected graph on n vertices, and let $S(G)$ be the set of all real symmetric $n \times n$ matrices whose nonzero off-diagonal entries occur in exactly the positions corresponding to the edges of G. The inverse eigenvalue problem of a graph (IEPG) asks: Given a graph G on n vertices and real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$, is there a matrix in $S(G)$ with eigenvalues equal to $\lambda_1, \lambda_2, \ldots, \lambda_n$? We also consider an extended version of the problem in which the eigenvalues of the matrix and a proper principal submatrix are prescribed.

We discuss three illuminating instances of these problems: G is the complete graph on n vertices, G is the path on n vertices, and G is one of the six connected graphs on 4 vertices. To do so we make use of $\mathrm{mr}(G)$, the minimum rank of all matrices in $S(G)$ and $M(G)$, the maximum nullity of all matrices in $S(G)$.

A plausible conjecture initially is that for G connected, if $\lambda_1, \lambda_2, \ldots, \lambda_n$ can be realized by a matrix in $S(G)$ and H is a graph obtained by inserting an edge into G, then $\lambda_1, \lambda_2, \ldots, \lambda_n$ can be realized by a matrix in $S(H)$. However, there is a simple counterexample with $n = 5$. The strong spectral property rehabilitates this conjecture with the result: Given a graph G on n vertices and a matrix $A \in S(G)$ that satisfies the strong spectral property, then for any supergraph H of G on n vertices, there is a matrix $B \in S(H)$ with the same eigenvalues as A.

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