Distinct Partial Sums in Cyclic Groups

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Let \((G, +)\) be an abelian group and consider a subset \(A \subseteq G\) with \(|A| = k\). Given an ordering \((a_1, \ldots, a_k)\) of the elements of \(A\), define its \textit{partial sums} by \(s_0 = 0\) and \(s_j = \sum_{i=1}^{j} a_i\) for \(1 \leq j \leq k\). Alspach conjectured that for any cyclic group \(\mathbb{Z}_n\) and any subset \(A \subseteq \mathbb{Z}_n \setminus \{0\}\) with \(s_k \neq 0\), it is possible to find an ordering of the elements of \(A\) such that no two of its partial sums \(s_i\) and \(s_j\) are equal for \(0 \leq i < j \leq k\).

We address this conjecture (and a weakening of it due to Archdeacon) in the case that \(n\) is prime and do the following. We show how Noga Alon’s Combinatorial Nullstellensatz can be used to frame the conjecture. Further, in the case that \(n\) is prime, we verify computationally that Alspach’s Conjecture is true for small values of \(|A|\). In the case that \(n\) is prime, we show that a sequence of length \(k\) having distinct partial sums exists in any subset of \(\mathbb{Z}_n \setminus \{0\}\) of size at least \(2k - \sqrt{8k}\) in all but at most a bounded number of cases.

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