Embedding Factorizations for 4-uniform Hypergraphs

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Let \( \binom{X}{h} \) be the collection of all \( h \)-subsets of an \( n \)-set \( X \supseteq Y \). A coloring (partition) of \( S \) is \( r \)-regular if the number of times each element of \( X \) appears in each color class (all sets of the same color) is the same number \( r \). Given an \( r \)-regular coloring of \( S \subseteq \binom{X}{h} \), we are interested in finding conditions under which this coloring is extendible to an \( s \)-regular coloring of \( \binom{X}{h} \). The case \( h = 2, r = 1 \) is extensively studied in the literature and is closely related to completing partial symmetric Latin squares, but very little is known for \( h \geq 3 \).

The case \( S = \emptyset, s = 1 \) was studied by Sylvester in the 18th century, and remained open until the 1970s.

In this paper we completely solve the case \( S = \binom{Y}{h}, h = 4 \). This settles the first open case of recent problem of Bahmanian and Newman. These results can be seen as extensions of the famous Baranyai’s theorem, and make progress toward settling a 40-year-old problem posed by Cameron.

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