Rényi α-Entropy of Tournaments

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The *Rényi α-entropy* of a discrete probability distribution $P = \{p_1, \ldots, p_n\}$, defined as $H_\alpha(P) = \frac{1}{1-\alpha} \log_2 \left(\sum_{p \in P} p^\alpha \right)$, is a generalization of the classical Shannon information entropy $H(P) = -\sum_{p_i \in P} p_i \log_2(p_i)$. Suppose G is an n-vertex undirected graph, and $L(G)$ is the Laplacian of G, but normalized so that the trace of $L(G)$ is equal to 1. The multiset of eigenvalues $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ of $L(G)$, each being nonnegative, real, and all summing to 1, may be regarded as a discrete probability distribution and, via $H(\Lambda)$, we obtain (what some call the von-Neumann) entropy of G. The functional H on the set of undirected graphs has been investigated recently by many, but if D is a directed graph and $L(D)$ is D's normalized Laplacian, the eigenvalues of $L(D)$ are not necessarily real and so the notion of Shannon (von-Neumann) entropy cannot be applied. We get around this by applying the Rényi α-entropy to investigate the entropy of directed graphs, and focus our investigation on tournaments (orientations of complete graphs). We determine which tournaments optimize the Rényi α-entropy for various α, and essentially investigate what the functional H_α says about the tournament to which it is applied. We observe that, as we increase α, H_α partitions the set of tournaments into a more and more refined set of equivalence classes. For example, if Q is a quadratic residue tournament (a regular tournament that is vertex- and arc-transitive) and R is a regular tournament that is not a quadratic residue tournament, $H_4(Q) > H_4(R)$. More generally H_α yields a weak ordering of the set of tournaments on n vertices, for some n and α; and a linear order for other n and α.

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