Rényi α-Entropy of Tournaments

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The Rényi α-entropy of a discrete probability distribution \( P = \{p_1, \ldots, p_n\} \), defined as

\[
H_\alpha(P) = \frac{1}{1 - \alpha} \log_2 \left( \sum_{p \in P} p^\alpha \right),
\]

is a generalization of the classical Shannon information entropy \( H(P) = -\sum_{p_i \in P} p_i \log_2(p_i) \).

Suppose \( G \) is an \( n \)-vertex undirected graph, and \( L(G) \) is the Laplacian of \( G \), but normalized so that the trace of \( L(G) \) is equal to 1. The multiset of eigenvalues \( \Lambda = \{\lambda_1, \ldots, \lambda_n\} \) of \( L(G) \), each being nonnegative, real, and all summing to 1, may be regarded as a discrete probability distribution and, via \( H(\Lambda) \), we obtain (what some call the von-Neumann) entropy of \( G \). The functional \( H \) on the set of undirected graphs has been investigated recently by many, but if \( D \) is a directed graph and \( L(D) \) is \( D \)'s normalized Laplacian, the eigenvalues of \( L(D) \) are not necessarily real and so the notion of Shannon (von-Neumann) entropy cannot be applied. We get around this by applying the Rényi α-entropy to investigate the entropy of directed graphs, and focus our investigation on tournaments (orientations of complete graphs). We determine which tournaments optimize the Rényi α-entropy for various \( \alpha \), and essentially investigate what the functional \( H_\alpha \) says about the tournament to which it is applied. We observe that, as we increase \( \alpha \), \( H_\alpha \) partitions the set of tournaments into a more and more refined set of equivalence classes. For example, if \( Q \) is a quadratic residue tournament (a regular tournament that is vertex- and arc-transitive) and \( R \) is a regular tournament that is not a quadratic residue tournament, \( H_4(Q) > H_4(R) \). More generally \( H_\alpha \) yields a weak ordering of the set of tournaments on \( n \) vertices, for some \( n \) and \( \alpha \); and a linear order for other \( n \) and \( \alpha \).

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