Combinatorics at the USA Mathematical Olympiad

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The American Mathematics Competitions program of the Mathematical Association of America consists of a series of examinations for middle school, high school, and college students, designed to build problem solving skills, foster a love of mathematics, and identify and nurture talented students across the United States. While our college competition consists of a single exam (the Putnam Exam), the high school competition (which has over 300,000 participants) has three rounds, the final one being the USA Mathematical Olympiad (USAMO). This competition follows the style of the International Mathematics Olympiad (in fact, it serves as one of the selection exams for the USA team for the IMO): it consist of three problems each on two consecutive days, with an allowed time of four and a half hours both days.

The problems of the USAMO are chosen – from a large collection of proposals submitted for this purpose – by the USAMO Editorial Board, whose co-editors-in-chief in 2020 and 2021 were Evan Chen and Jennifer Iglesias, with associate editors Ankan Bhattacharya, John Berman, Zuming Feng, Sherry Gong, Alison Miller, Maria Monks Gillespie, and Alex Zhai.

Here we feature four problems from the past two years that are in combinatorics or have some combinatorial flavor. The solutions presented are those of the present author, relying in part on the submissions of the problem authors or the contestants. For each problem, we also include the mean scores of the contestants; these ranged from 0.74 to 4.14 on a scale of $[0, 7]$.

I hope you enjoy these beautiful problems! You may want to spend a bit of time with them before reading the solutions; for this reason, the problems and their solutions are separated to different pages. Please get in touch if you wish to get involved with the USA Mathematical Olympiad or with one of our other competitions.

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Problems

Problem 1 (2020 USAMO 2, proposed by Alex Zhai; mean score of 4.14).

An empty $2020 \times 2020 \times 2020$ cube is given, and a 2020×2020 grid of square unit cells is drawn on each of its six faces. A *beam* is a $1 \times 1 \times 2020$ rectangular prism. Several beams are placed inside the cube subject to the following conditions:

- The two 1×1 faces of each beam coincide with unit cells lying on opposite faces of the cube. (Hence, there are $3 \cdot 2020^2$ possible positions for a beam.)
- No two beams have intersecting interiors.
- The interiors of each of the four 1×2020 faces of each beam touch either a face of the cube or the interior of the face of another beam.

What is the smallest positive number of beams that can be placed to satisfy these conditions?

Problem 2 (2020 USAMO 4, proposed by Antonia Bluher and Richard Stong; mean score of 3.12).

Suppose that $(a_1, b_1), (a_2, b_2), \ldots, (a_{100}, b_{100})$ are distinct ordered pairs of nonnegative integers. Let N denote the number of pairs of integers (i, j) satisfying $1 \leq i < j \leq 100$ and $|a_i b_j - a_j b_i| = 1$.

Determine the largest possible value of N over all possible choices of the 100 ordered pairs.

Problem 3 (2021 USAMO 2, proposed by Zoran Sunic; mean score of 2.61).

The Planar National Park is a subset of the Euclidean plane consisting of several trails which meet at junctions. Every trail has its two endpoints at two different junctions, whereas each junction is the endpoint of exactly three trails. Trails only intersect at junctions (in particular, trails only meet at endpoints). Finally, no trails begin and end at the same two junctions. (An illustartion is shown in Figure 1 below.)

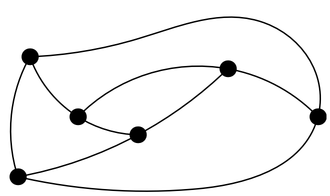

Figure 1: Illustration for Problem 3

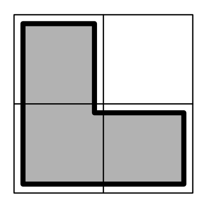

Figure 2: Illustration for Problem 4

A visitor walks through the park as follows: she begins at a junction and starts walking along a trail. At the end of that first trail, she enters a junction and turns left. On the next junction she turns right, and so on, alternating left and right turns at each junction. She does this until she gets back to the junction where she started.

What is the largest possible number of times she could have entered any junction during her walk, over all possible layouts of the park?

Problem 4 (2021 USAMO 3, proposed by Shaunak Kishore and Alex Zhai; mean score of 0.74).

Let $n \geq 2$ be an integer. An $n \times n$ board is initially empty. Each minute, you may perform one of three moves:

- If there is an L-shaped tromino region of three cells without stones on the board (see Figure 2; rotations not allowed), you may place a stone in each of those cells.
- If all cells in a column have a stone, you may remove all stones from that column.
- If all cells in a row have a stone, you may remove all stones from that row.

For which n is it possible that, after some nonzero number of moves, the board has no stones?

Solutions

Problem 1. Let n be a positive even integer, and consider an $n \times n \times n$ cube. We claim that the smallest positive number of beams satisfying the three analogous conditions (where 2020 is replaced by n) is $3n/2$ (and thus equals 3030 for the case of $n = 2020$).

To facilitate our deliberation, we place the cube into the Cartesian coordinate system so that its edges are parallel to the coordinate axes and that one of its corners is at the origin and another at the point (n, n, n). We can then label the n^3 cells of the cube by their corners furthest from the origin. For positive integers $a, b \leq n$, we then let $B_x(a, b)$ denote the *type x beam* consisting of cells (t, a, b) with $t = 1, 2, \ldots, n$; similarly, we let $B_y(a, b)$ denote the *type y beam* consisting of cells (a, t, b), and $B_z(a, b)$ denote the *type z beam* consisting of cells (a, b, t).

To see that there is a valid configuration with $3n/2$ beams, consider the collection consisting of beams

$$B_x(1, 1), B_x(3, 3), \ldots, B_x(n-1, n-1),$$

$$B_y(1, n), B_y(3, n-2), \ldots, B_y(n-1, 2),$$

and

$$B_z(2, 2), B_z(4, 4), B_z(6, 6), \ldots, B_z(n, n),$$

illustrated here for $n = 10$.

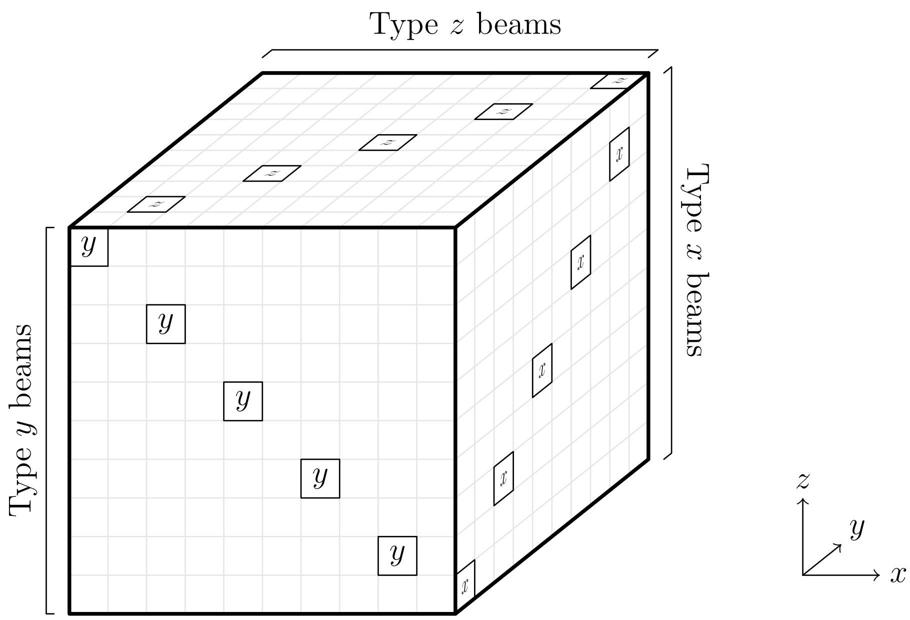

Figure 3: An illustration of the placements of the beams for $n = 10$

Clearly, our collection consists of $3n/2$ beams satisfying the first requirement. We can verify that our collection also satisfies the second condition, as follows. Note that

- each type x beam in the collection consists of cells whose second and third coordinates are odd,
- each type y beam consists of cells whose first coordinate is odd and third coordinate is even, and
- each type z beam consists of cells whose first and second coordinates are even.

Therefore, no two beams in the collection have intersecting interiors, and thus the second required condition holds.

Finally, to see that our third condition is satisfied, consider first a type x beam B. We see that the interior of each horizontal face (that is, each face perpendicular to the z axis) of B touches the interior of a type y beam or the a face of the cube, and the interior of each vertical face (that is, each face perpendicular to the y axis) of B touches the interior of a type z beam or the a face of the cube. Since analogous observations apply to type y beams and type z beams, we conclude that our collection of beams satisfies all three required conditions.

It remains to be shown that any valid construction must use at least $3n/2$ beams. Let N_x, N_y, and N_z denote the number of beams of type x, y, and z, respectively. If any two of these three quantities are zero, then the third must equal n^2, since the collection must then contain all possible beams of that type. As $n^2 > 3n/2$, we may assume that at least two of N_x, N_y, or N_z are nonzero.

Next, we prove that $N_x + N_y \geq n$. By a *horizontal beam*, we mean a beam of type x or type y; by the previous paragraph, we must have at least one horizontal beam. According to the third condition, the interior of each horizontal face of each horizontal beam must touch the face of the cube or the interior of a face of another beam; clearly, this other beam would also need to be a horizontal beam. Repeating this observation, we find that we must have at least n horizontal beams, proving our claim.

By a similar argument, we have $N_y + N_z \geq n$ and $N_z + N_x \geq n$, and so

$$N_x + N_y + N_z = \frac{1}{2}(N_x + N_y) + \frac{1}{2}(N_y + N_z) + \frac{1}{2}(N_z + N_x) \geq \frac{3}{2}n,$$

proving our claim. □

Problem 2. *First Solution.* We claim that the answer is $N = 2n - 3$ for $n \geq 2$ ordered pairs (197 for $n = 100$). Let $P_1 = (a_1, b_1), \ldots, P_n = (a_n, b_n)$. We say that points P_i and P_j are *enchanted* if $|a_i b_j - a_j b_i| = 1$; note that, by the Shoelace Formula, this is equivalent to triangle OP_iP_j (where O is the origin) having area $1/2$. It is easy to see that the n points $P_1 = (0, 1), P_2 = (1, 2), P_3 = (1, 3), \ldots, P_n = (1, n)$ contain $2n - 3$ enchanted pairs: P_1 is enchanted with the other $n - 1$ points, and P_i is enchanted with P_{i+1} for each $i = 2, 3, \ldots, n - 1$.

We will now use induction to prove that $N \leq 2n - 3$ for every $n \geq 2$. This being trivial for $n = 2$, assume that our claim holds for each collection of $n - 1$ points for some $n \geq 3$, and consider a collection $P_1 = (a_1, b_1), \ldots, P_n = (a_n, b_n)$. Without loss of generality, we assume that $a_n + b_n \geq a_i + b_i$ for all $1 \leq i \leq n$. By our inductive assumption, it suffices to show that P_n is enchanted with at most two other points.

If this were not the case, then we would have two points P_i and P_j that have the same distance from line OP_n and are on the same side of that line. But then P_iP_j is parallel to OP_n, so $\overrightarrow{P_iP_j} = t \cdot \overrightarrow{OP_n} = \langle ta_n, tb_n \rangle$ for some scalar t, and we may assume that $t > 0$. Since P_i and P_j have integer coordinates, ta_n and tb_n are integers, so $|b_i ta_n - a_i tb_n| = t$ is an integer as well. Therefore, $t \geq 1$, and since $P_i \neq O$, we arrive at $a_j + b_j = (a_i + ta_n) + (b_i + tb_n) > a_n + b_n$, contradicting our choice of P_n. □

Second Solution. First, we recall that the *Farey sequence* of order m is the increasing sequence of fractions a/b of relatively prime integers a and b with $0 \leq a \leq b \leq m$. The property of Farey sequences that we need here is that a/b and a'/b' are consecutive terms in some Farey sequence if, and only if, $|a'b - ab'| = 1$.[2]

Suppose now that $(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)$ are distinct ordered pairs of nonnegative integers, so that there are N pairs of integers (i, j) satisfying $1 \leq i < j \leq n$ and $|a_i b_j - a_j b_i| = 1$. We shall use induction to prove that $N \leq 2n - 3$ for every $n \geq 2$. This obviously holds for $n = 2$ and $n = 3$, so let $n \geq 4$. Without loss of generality, we arrange our points so that $\max(a_n, b_n) \geq \max(a_i, b_i)$ for each $1 \leq i \leq n$; furthermore, we may assume that $a_n \leq b_n$, since if this were not the case, we could instead consider the mirror image of our n points with respect to the line $y = x$. Note that our assumptions imply that $b_n \geq 2$, since we can only have $b_n = 1$ for $n \leq 3$. Our goal is to show that there are at most two indices $1 \leq i \leq n - 1$ for which $|a_i b_n - a_n b_i| = 1$; our claim will then follow by induction.

Observe that if $|a_i b_n - a_n b_i| = 1$, then $a_i \leq b_i$. Indeed, if this were not the case, then we would have $a_i \geq b_i + 1$, so

$$1 = |a_i b_n - a_n b_i| = a_i b_n - a_n b_i \geq (b_i + 1)b_n - a_n b_i = b_i(b_n - a_n) + b_n \geq b_n \geq 2,$$

a contradiction. Note also that $|a_i b_n - a_n b_i| = 1$ implies that $b_i \neq 0$ and $\gcd(a_i, b_i) = \gcd(a_n, b_n) = 1$. Therefore, a_i/b_i and a_n/b_n are consecutive terms in some Farey sequence; we need to show that this can happen for at most two indices i. But this is clearly the case: a_n/b_n appears only in Farey sequences of order b_n or more; it can have at most two neighbors in the

[2]For this and other interesting features of Farey sequences see, for example, Chapter III in An introduction to the theory of numbers, by G. H. Hardy and E. M. Wright, Sixth edition, *Oxford University Press, Oxford*, 2008.

Farey sequence of order b_n; and if a_i/b_i and a_n/b_n are not consecutive in the Farey sequence of order b_n, then they are also not consecutive in one with order more than b_n.

To show that $N = 2n-3$ is achievable, we may start with any two fractions a/b and a'/b' that are consecutive in some Farey sequence; inserting $(a+a')/(b+b')$ between them results in three consecutive fractions, since $|b(a+a') - a(b+b')| = 1$ and $|b'(a+a') - a'(b+b')| = 1$. Repeating this process then yields n points with $N = 2n-3$ pairs with the desired property. (For example, starting with $0/1$ and $1/2$, and inserting the just-described fraction next to $0/1$ each time, yields the sequence $0/1, 1/n, 1/(n-1), \dots, 1/2$, corresponding to the example of the first solution.) □

Problem 3. The answer is three times. We begin by exhibiting an example of a park layout which features three visits. Sketched in Figure 4 is one of many possible constructions. The path starts from C and walks toward A, and continues as follows:

$$C \to A \to H \to I \to F \to G \to D \to B \to A \to H \to E \to F \to G \to J \to B \to A \to C$$

As we see, this path visits A three times.

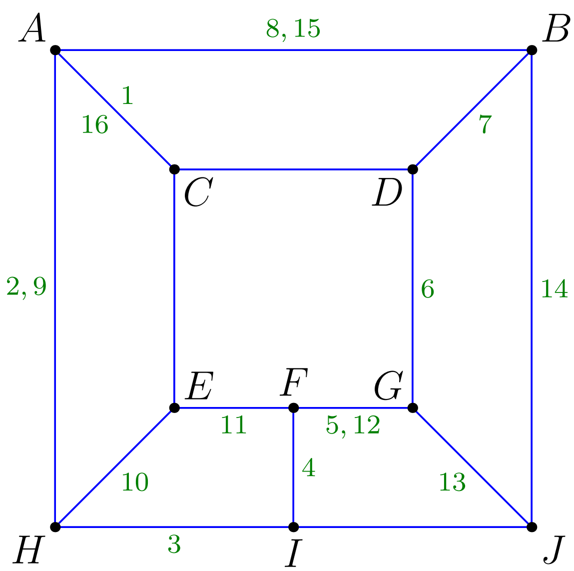

Figure 4: An example achieving three visits

We will prove that the visitor cannot visit any junction more than three times. (This trivially holds fore the initial/terminal junction.) Note that if a junction were to be visited four times or more, then this would mean four or more arrivals and four or more departures, which is only possible if at least one of the three trails meeting at that junction would have had to be traversed at least three times. Therefore, it suffices to show that no trail can be on the visitor's path more than twice.

Suppose, indirectly, that there is a trail that the visitor walked on three or more times. (Note that this trail cannot be adjacent to the junction where her walk started.) This then implies that at least two of those times she turned in the same direction (left or right) when she reached the end of the trail. Let's assume then that her m-th and n-th trail during her walk is the same for some $2 \le m < n$ with the same turn at the end; we may further assume that this is the trail with the smallest possible m. Let A and B denote the junctions at the two ends of this trail.

Now if she walked along this trail both times in the same direction, say from A to B, and made the same turn at the end (e.g., left), then her $(m-1)$-st and $(n-1)$-st trails were also the same, and she made the same turn when she got to A (right). This contradicts the minimality of m. On the other hand, if once she walked from A to B and then later from B to A, but turning in the same direction at the end both times (e.g., left), then her $(m-1)$-st trail was the same as her $(n+1)$-st, and she made the same turn at the ends of these two trails as well (right), again contradicting the minimality of m. This completes our proof. □

Problem 4. We claim that the answer is all positive integers n that are divisible by 3. First, we show that the procedure is possible in each of these cases.

When n is divisible by 3, one may divide the board into 3×3 sub-squares; for brevity, let us refer to these $(n/3)^2$ sub-squares as *cages*. We then follow the procedure illustrated in Figure 5, as follows.

- First, we put two non-overlapping L-trominoes in each cage, as shown in the first step.
- This causes every center column of each cell to be completely filled. Thus, we may remove all $n/3$ columns which correspond to the center columns of cages, as shown in the second step.
- In each cage, we then place one L-tromino as shown in the third step.
- Now the board consists of $2n/3$ completely filled rows, so we may eliminate them all.

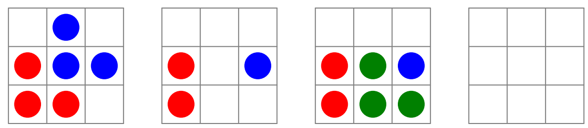

Figure 5: The four-step procedure that clears all stones

We now prove that if after some sequence of moves no stones remain, then n must be a multiple of 3. We will employ what is usually called the *polynomial method*; in particular, we make use of the following.

Lemma. *Consider the polynomial*

$$f(x,y) = \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} d_{i,j} x^i y^j,$$

where the coefficients $d_{i,j}$ are real numbers and $d_{n_1,n_2} \neq 0$. If A_1 and A_2 are sets of real numbers with $|A_1| > n_1$ and $|A_2| > n_2$, then there are elements $a_1 \in A_1$ and $a_2 \in A_2$ for which $f(a_1, a_2) \neq 0$.[3]

Let us introduce some notations. We parametrize the cells of the board by letting (i,j) denote the position of the cell in the i-th column (counting from the left) and the j-th row (counting from the bottom). We then associate each state of the board with the polynomial

$$A(x,y) = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{i,j} x^{i-1} y^{j-1},$$

where $c_{i,j}$ is 1 when there is a stone in cell (i,j) and 0 otherwise. This allows us to think of the chain of moves as the sequence $(A_k(x,y))_{k=0}^{m}$ where $A_0(x,y) = 0$ (representing the initial position of the board), $A_m(x,y) = 0$ (expressing the fact that there are no stones on the board after m moves for some $m \in \mathbb{N}$), and where $A_k(x,y)$ results from $A_{k-1}(x,y)$ in one of the following ways:

- $A_k(x,y) = A_{k-1}(x,y) + x^{i-1}y^{j-1}(1+x+y)$ if a tromino was placed on the board with its lower left corner at position (i,j);
- $A_k(x,y) = A_{k-1}(x,y) - x^{i-1}(1+y+y^2+\cdots+y^{n-1})$ if all stones got removed from column i; and
- $A_k(x,y) = A_{k-1}(x,y) - y^{j-1}(1+x+x^2+\cdots+x^{n-1})$ if all stones got removed from row j.

We need a few more notations. For $i = 1,2,\ldots,n-1$ and $j = 1,2,\ldots,n-1$, we let $a_{i,j}$ denote the number of times a tromino was added with its lower-left corner at position (i,j); we then set

$$P(x,y) = \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} a_{i,j} x^{i-1} y^{j-1}.$$

Furthermore, we set $r(j)$ and $c(i)$ equal to the number of times the j-th row and i-th column were cleared, respectively, and define $Q(x) = \sum_{i=1}^{n} c_i x^{i-1}$ and $R(y) = \sum_{j=1}^{n} r_j y^{j-1}$. With these notations, the fact that our procedure succeeded can be stated by the equation

$$P(x,y)(1+x+y) - Q(x)(1+y+y^2+\cdots+y^{n-1}) - R(y)(1+x+x^2+\cdots+x^{n-1}) = 0.$$

[3]Note that this lemma is the two-variable version of the well-known fact that a nonzero polynomial cannot have more roots than its degree. For a simple proof and a variety of applications see, for example, Section 12.3 in Number Theory, by R. Freud and E. Gyarmati, *American Mathematical Society, Providence*, 2020.

Take A to be the set of n-th roots of unity other than 1; that is, the $n-1$ distinct complex numbers a for which

$$\frac{a^n - 1}{a - 1} = 1 + a + a^2 + \cdots + a^{n-1} = 0.$$

Since $P(x, y)$ is a nonzero polynomial with x-degree and y-degree at most $n-2$, our lemma above guarantees elements $a_1, a_2 \in A$ for which $P(a_1, a_2) \neq 0$. But since substituting $x = a_1$ and $y = a_2$ into our equation yields

$$P(a_1, a_2)(1 + a_1 + a_2) = 0,$$

this can only occur when $1 + a_1 + a_2 = 0$. Therefore, the imaginary parts of a_1 and a_2 are negatives of one another; recalling that both numbers have norm 1, this then implies that their real parts have the same absolute value. But these real parts must then both be negative, and in fact equal to $-\frac{1}{2}$, so a_1 and a_2 are $-\frac{1}{2} \pm \frac{\sqrt{3}}{2}\mathrm{i}$. We thus got that a_1 and a_2 are third roots of unity. However, this can only happen if n is divisible by 3: indeed, if $n = 3q + r$ for some integers q and r with $r \in \{0, 1, 2\}$, then $a_1^r = a_1^n / a_1^{3q} = 1$, which is only possible when $r = 0$, as claimed. $\square$