Graphs on groups

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Abstract

The topic of my talk began life in 1955 with a paper by Brauer and Fowler which, in hindsight, was the first step on the thousand-mile journey to the Classification of Finite Simple Groups. The authors of the paper used what is now called the *commuting graph* of the group, whose vertices are the group elements, two vertices x and y joined if they commute (that is, $xy = yx$). By examining distances in the graph (with the identity removed), they were able to prove very strong results about finite simple groups. Curiously, the word "graph" never appears in the paper; the distance is defined in terms of the commuting relation.

Since then, a number of other graphs on a group have been defined, including the power graph, enhanced graph, and nilpotence graph. In another recent variant, given an equivalence relation on the group, we can look at the qoutient graph.

In summer 2021, when international travel was out of the question, I helped run a virtual research discussion group in Kochi, Kerala, in south India. This was very successful, introducing a lot of researchers to the problems of graphs on groups, and has resulted in a number of significant results, some of which I will discuss. The graphs form a hierarchy, and some of the most interesting results arise from comparing different graphs in the hierarchy – the groups for which two of these graphs are equal include a lot of well-studied classes of groups. Other questions concern universality (which graphs can be embedded as induced subgraphs in a particular type of graph for some group), and clique number (since cliques often have group-theoretic significance).

I will survey some of the best results that have been obtained.