The geometry of diagonal groups

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Abstract

Diagonal groups are a class of permutation groups which first appeared in the O'Nan–Scott Theorem describing the structure of primitive permutation groups. But in fact they form a much wider class, and can be defined for any group, finite or infinite, and not just for finite simple groups.

In projective geometry we meet the fact that projective planes exist in great profusion, while higher dimensional projective spaces are much more restricted (and in fact are coordinatised by an algebraic structure which emerges from the geometry). Our main theorem is analogous to this. Diagonal groups are automorphism groups of structures we call *diagonal semilattices* (they are join semilattices of the partition lattice). Two-dimensional diagonal semilattices are essentially the same as Latin squares, and exist in great profusion; but in higher dimensions, things are much tighter, and the structures are coordinatised by an algebraic structure (a group) which emerges naturally from the combinatorics.

There are further aspects of the theory. The notion of diagonal semilattice can be generalised in a similar way to the extension from Latin squares to sets of mutually orthogonal Latin squares; we have started investigating this but much is not yet known. There is also a graph associated with such a structure, which generalizes the classes of strongly regular Latin square graphs and of distance-transitive folded cubes; we have a conjecture about the chromatic number of these graphs which would extend the well-known Hall–Paige conjecture about Latin squares (proved in 2009).