Analysis Qualifying Exam
January 21, 2016

INSTRUCTIONS

- Number all your pages and write only on one side of the paper. Anything written on the second side of a page will be ignored.
- Write your name at the top of each page.

1. Let $f : \mathbb{R} \to \mathbb{R}$ satisfy $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$ and $f(1) = 1$. Prove:
   (a) $f(x) = x$ for all rational numbers $x$.
   (b) Assume, in addition, that $f$ is continuous at 0. Prove that then $f(x) = x$ for all $x \in \mathbb{R}$.

2. We say that a family of subsets of a metric space $(X, d)$ is locally finite if for each $p \in X$ there is an open set $V$ such that $p \in V$ and $V$ only intersects a finite number of the sets $F_n$. Prove: If $\{F_n\}$ is a locally finite family of closed sets, then $\bigcup_{n=1}^{\infty} F_n$ is closed.

3. Let $X$ be the metric space consisting of all sequences $a = (a_1, a_2, \ldots)$ of real numbers such that $\sum_{n=1}^{\infty} |a_n| < \infty$, with the distance function defined by
   \[ d(a, b) = \sum_{n=1}^{\infty} |a_n - b_n| \]
   if $a = (a_1, a_2, \ldots), b = (b_1, b_2, \ldots)$.
   (a) Prove $B(0, 1) = \{a = (a_1, a_2, \ldots) : d(a, 0) \leq 1\}$ is not compact.
   (b) Let $C = \{a = (a_1, a_2, \ldots) : |a_n| \leq 1/n^2$ for $n \in \mathbb{N}\}$. Prove $C$ is compact.

4. Let $X$ be a metric space and let $f_n : X \to \mathbb{R}$ for each $n \in \mathbb{N}$. We say that the sequence $\{f_n\}$ is locally uniformly convergent if for every $p \in X$ there exists an open set $U$ in $X$ such that $p \in U$ and the sequence of restrictions $\{f_n|_U\}$ converges uniformly on $U$. Prove: If $X$ is compact and the sequence $\{f_n\}$ converges locally uniformly, then it is uniformly convergent.

5. Let $f : [-1, 1] \to \mathbb{R}$ be continuous and even ($f(-x) = f(x)$ for all $x \in [-1, 1]$). Prove: For each $\epsilon > 0$ there exists a polynomial $p$ such that $|f(x) - p(x^2)| < \epsilon$ for all $x \in [-1, 1]$.

6. Prove or disprove: There exists a closed subset $F$ of $\mathbb{R}$ such that $F$ has positive measure and $F \cap \mathbb{Q} = \emptyset$.

7. Evaluate, justifying all steps:
   \[ \lim_{n \to \infty} \int_{0}^{\infty} \frac{n \sin \frac{x}{n}}{x(1 + x^2)} \, dx. \]
   **Hint:** You may use that $|\sin x/x| \leq 1$ for all $x \in (0, \infty)$. 
1. Let \( u_n \geq 0 \) for all \( n \in \mathbb{N} \). **Prove:** If \( \sum_{n=1}^{\infty} u_n \) converges, then \( \sum_{n=1}^{\infty} \frac{\sqrt{u_n}}{n} \) also converges.

2. **Prove** there exists a unique differentiable function \( \Phi : \mathbb{R} \rightarrow \mathbb{R} \) such that \( \Phi'(x) = e^{-x^2} \) for all \( x \in \mathbb{R} \) and \( \Phi(0) = 0 \).

3. Let \( X \) be a metric space, let \( C \subseteq X \) have the property that if \( x, y \in C \), there exists a connected subset \( A \) of \( C \) such that \( x, y \in A \). **Prove:** \( C \) is connected.

4. Let \( E \) be an equicontinuous and bounded set of functions from \([0, 1]\) to \( \mathbb{R} \). **Prove:** If \( \{f_n\} \) is a sequence in \( E \) that converges for each **rational** \( x \in [0, 1] \), then \( \{f_n\} \) converges uniformly.

5. Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) be Lebesgue integrable. **Prove** that
\[
\lim_{n \to \infty} \int_{\mathbb{R}} f(x) \cos nx \, dx = 0.
\]

6. Assume \( f : \mathbb{R} \rightarrow \mathbb{R} \) is continuous. **Prove:** The inverse image under \( f \) of a Borel set is a Borel set.

7. **Prove** that the following limit exists:
\[
\lim_{n \to \infty} \int_{0}^{\infty} \frac{\cos x}{nx^2 + 1/n} \, dx.
\]

Be sure to justify all steps.

**Hint:** Change variables by \( t = nx \).
1. Let $\{a_n\}$ be a sequence of real numbers that converges to $a$. Show that
\[
\lim_{n \to \infty} \frac{a_1 + a_2 + \cdots + a_n}{n} = a.
\]

2. Consider the series
\[
\sum_{n=1}^{\infty} \frac{x}{(1 + x)^n}.
\]
Show that the series converges for all $x \geq 0$ and that it converges uniformly on the interval $[r, \infty)$ for every $r > 0$. Does the series converge uniformly on $(0, \infty)$? Justify your answer.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function that satisfies, for each $x \in \mathbb{R}$,
\[
f(x) \leq \frac{f(x-h) + f(x+h)}{2}
\]
for all $h > 0$. Show that the maximum value of $f$ on any bounded closed interval $[a, b]$ is attained at one of the endpoints, that is, either $f(a)$ or $f(b)$ is the maximum value of $f$ on the interval $[a, b]$.

4. (a) Show that the inequalities
\[
\frac{x}{x+1} < \ln(1+x) < x
\]
hold for all $x > 0$.

(b) Define
\[
f(x) = \left(1 + \frac{1}{x}\right)^x
\]
and
\[
g(x) = \left(1 + \frac{1}{x}\right)^{x+1}
\]
for all $x > 0$, where we define $x^y = e^{y \ln x}$ for all $x > 0$ and $y > 0$ and $e$ is Euler’s number (also known as Napier’s constant). Show that $f$ is strictly increasing while $g$ is strictly decreasing on the interval $(0, \infty)$, and that $f(x) < e < g(x)$ for all $x > 0$.

5. Let $\{f_n\}$ be a sequence of real-valued functions defined on a compact metric space $(X, d)$ such that $f_n(x_n) \to f(x)$ in $\mathbb{R}$ whenever $x_n \to x$ in $X$. Assume that $f$ is continuous. Show that $\{f_n\}$ converges uniformly to $f$.

6. (a) Show that the sequence of functions
\[
f_n(x) = e^{-n(ax-1)^2}
\]
point-wise converges to zero but not uniformly on $[0, 1]$.

(b) Nevertheless show that
\[
\lim_{n \to \infty} \int_0^1 f_n(x) \, dx = 0.
\]
7. Let \((X,d)\) be a compact metric space. Assume that \(f : X \to X\) is an expansion map, that is, \(d(f(x), f(y)) \geq d(x, y)\) for all \(x, y \in X\). For every \(x \in X\), define \(f^2(x) = f(f(x))\), \(f^3(x) = f(f^2(x))\), and in general \(f^n(x) = f(f^{n-1}(x))\) for \(n \geq 2\). Prove the following statements:

(a) For every \(x \in X\), we have \(d(x, f^{m-n}(x)) \leq d(f^n(x), f^m(x))\) for all positive integers \(m\) and \(n\) with \(m > n\), and that the sequence \(\{f^n(x)\}\) contains a subsequence \(\{f^{n_k}(x)\}\) such that \(f^{n_k}(x) \to x\) as \(k \to \infty\).

(b) For every pair of points \((x, y)\), the sequence \(\{f^n\}\) contains a subsequence \(\{f^{n_k}\}\) such that \(f^{n_k}(x) \to x\) and \(f^{n_k}(y) \to y\) as \(k \to \infty\). (Hint: consider the compact metric space \(X \times X\) and the product metric \(D((x, y), (u, v)) = d(x, u) + d(y, v)\) and the map \(F : X \times X \to X \times X\) defined by \(F(x, y) = (f(x), f(y))\))

(c) For all \(x, y \in X\), \(d(f(x), f(y)) = d(x, y)\), that is, \(f\) is an isometry.
Analysis Qualifying Examination

In some cases partial credit may be given, but you should endeavor to fully complete as many problems as possible.

1. Let \( f \) be a real-valued differentiable function defined on \((-\infty, +\infty)\). Suppose that \( f \) has a bounded derivative. Show that there exist nonnegative constants \( A \) and \( B \) such that \( |f(x)| \leq A|x| + B \) for all \( x \in (-\infty, +\infty) \).

2. Consider the series
\[
\sum_{n=1}^{\infty} \frac{n^2 x^2}{1 + n^4 x^4}.
\]
(a) Show that for every \( \delta > 0 \) the series converges uniformly on the set \( \{ x : |x| \geq \delta \} \).
(b) Does the series converge uniformly on \( (-\infty, \infty) \)? Justify your answer.

3. Let \( f \) be a continuous real-valued function on \([a, b]\). Suppose that there exists a constant \( M \geq 0 \) such that
\[
|f(x)| \leq M \int_a^x |f(t)| \, dt
\]
for all \( x \in [a, b] \). Show that \( f(x) = 0 \) for all \( x \in [a, b] \).

4. Let \( A \) and \( B \) be two nonempty subsets of \( \mathbb{R}^n \), where \( \mathbb{R}^n \) is the \( n \)-dimensional Euclidean space equipped with the usual metric. Define \( A + B = \{ a + b : a \in A, b \in B \} \), where \( a + b \) is the sum of vectors \( a \) and \( b \) in the usual sense.
(a) If \( A \) and \( B \) are compact, show that \( A + B \) is compact.
(b) If \( A \) and \( B \) are connected, show that \( A + B \) is connected.
(b) If one of \( A \) and \( B \) is open, show that \( A + B \) is open.

5. Show that the following limit exists and find the limit.
\[
\lim_{n \to \infty} \int_0^\infty \frac{\cos(x^n)}{1 + x^n} \, dx.
\]

6. Let \( f \) be a real-valued measurable function defined on a bounded measurable set \( E \). Suppose that there exists a \( \delta > 0 \) such that \( \int_F |f| < 1 \) whenever \( F \) is a measurable subset of \( E \) and \( m(F) < \delta \). Show that \( f \) is Lebesgue integrable on \( E \). Here \( m \) denotes the Lebesgue measure.

7. Let \( E \) be a measurable subset of \( \mathbb{R} \). Define \( E^2 = \{ x^2 : x \in E \} \). If \( E \) has measure zero, show that \( E^2 \) also has measure zero.
[1] Let \((M, \rho)\) be a metric space. Suppose that \(f : M \to \mathbb{R}\) is uniformly continuous. Show that if \(\{x_n\}\) is a Cauchy sequence in \(M\), then the sequence \(\{f(x_n)\}\) is Cauchy in \(\mathbb{R}\).

Give an example which shows that the result is not necessarily true if \(f\) is assumed to be continuous but not uniformly continuous.

[2] Show that the set \(A = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}\) is a compact, connected subspace of the Euclidean space \(\mathbb{R}^2\).

[3] A subset \(A\) of a metric space \((M, \rho)\) is precompact if its closure \(\text{cl}(A)\) is compact. Show that if \(A\) is precompact, then for every \(\epsilon > 0\) there exists a finite covering of \(A\) by open balls of radius \(\epsilon\) with centers in \(A\).

[4] Let \(\epsilon > 0\). Construct an open subset \(S\) of \([0, 1]\) with Lebesgue measure less than \(\epsilon\) so that the closure of \(S\) is \([0, 1]\).

[5] Let \(\lambda\) be the Lebesgue measure on \(\mathbb{R}\). Suppose \(E\) is a Lebesgue measurable subset of \([0, 1]\) with \(\lambda(E) = 1\). Show that \(E\) is dense in \([0, 1]\).

[6] Let \(\{f_n\}_{n=1}^{\infty}\) be a sequence of Lebesgue measurable functions on a Lebesgue measurable subset \(E\) of \(\mathbb{R}\) which converges pointwise to a function \(f\). Suppose \(\int_E |f_n| \leq 1\) for all \(n \in \mathbb{N}\). Prove \(\int_E |f| \leq 1\).

[7] Compute
\[
\lim_{n \to \infty} n \int_0^1 \frac{\sin(u)}{u} e^{-nu} \sin(nu) \, du.
\]
Analysis Qualifying Exam.
Sept 6, 2013

Student Name (Print)______________________________

There are 7 questions.
1. Assume that \( \{x_k\} \) is a Cauchy sequence in a metric space, \((M, \rho)\), and that a subsequence, \( \{x_{k_n}\} \), converges. Prove that \( \{x_k\} \) converges.
2. Prove that if $0 < s \leq 1$ then $f(x) = x^s$ is uniformly continuous on $[0, \infty]$. 
3. Assume that \((\mathbb{M}, \rho)\) is a compact metric space, and that \(\{G_a\}\) is an open cover of \(\mathbb{M}\). Prove that there exists \(\delta > 0\) so that any ball with radius smaller than \(\delta\) is a subset of at least one of the \(G_a\).
4. Assume that $f$ and $g$ are continuous non-negative functions on a compact metric space, $(\mathbb{M}, \rho)$ and that \( \{x : g(x) = 0\} \subseteq \{x : f(x) = 0\} \). Prove that for any \( \varepsilon > 0 \) there exists $K(\varepsilon)$ so that for all $x \in \mathbb{M}$,

\[
f(x) \leq \varepsilon + K(\varepsilon)g(x).
\]
5. Assume that $(\mathbb{M}, \rho)$ is a complete metric space and that $f: \mathbb{M} \to \mathbb{M}$ is a uniform contraction, that is to say, that there exists $0 < q < 1$ so that $\forall x, y \in \mathbb{M}$,

$$\rho(f(x), f(y)) \leq q \rho(x, y)$$

Prove that there exists a unique $x \in \mathbb{M}$ so that $f(x) = x$. 

6. Assume that $f$ is a real-valued Lebesgue measurable function on $\mathbb{R}$ and for all $\alpha > 0$,
\[ \lambda(\{x:f(x) > \alpha\}) = \lambda(\{x:f(x) < -\alpha\}) \]
then
\[ \lambda(\{x:|f| > \alpha\}) \leq 2e^{-\alpha^2} \int_{-\infty}^{\infty} e^{a(x)} dx. \]

Note: $\lambda$ denotes Lebesgue measure on $\mathbb{R}$.
7. Prove that if \( f \) is a Lebesgue measurable function and
\[
\int_{-\infty}^{\infty} |f(x)| \, dx < \infty
\]
then \( \forall \varepsilon > 0 \) there exists a Lebesgue measurable set, \( E(\varepsilon) \), so that \( \lambda(E(\varepsilon)) < \infty \), \( |f| \) is bounded on \( E(\varepsilon) \), and
\[
\int_{(-\infty,\infty) \setminus E(\varepsilon)} |f(x)| \, dx < \varepsilon.
\]
1. Give a proof of Dini’s Theorem: let $F_n : [a, b] \to \mathbb{R}$ be an increasing sequence of continuous functions (i.e., $F_{n+1} \geq F_n$ for all $n$) converging pointwise to a continuous function $F$. Then $(F_n)$ converges to $F$ uniformly.

2. Let $f : \mathbb{R} \to \mathbb{R}^+$ be a continuous integrable function. Show that

$$f \text{ is uniformly continuous } \iff \lim_{|x| \to \infty} f(x) = 0$$

3. Let $f : (0, 1] \to \mathbb{R}$ be differentiable such that $f'$ is bounded on $(0, 1]$. Prove that $\lim_{x \to 0^+} f(x)$ exists.

4. Show that the series

$$\sum_{n \geq 1} \frac{x^n}{n+1}(1-x)$$

converges uniformly on $[-1, 1]$.

5. Let $g : \mathbb{R} \to \mathbb{R}$ be a measurable function. Assume that for each measurable set $B \subset \mathbb{R}$

$$\int_B g d\lambda = 0.$$  

Show that $g \equiv 0$ a.e.

6. Compute

$$\lim_{n \to \infty} \int_0^1 \frac{n^4u^2e^{-nu}du}{1+n^2u}$$
1. Suppose that each $f_n$ is increasing and continuous on $[a, b]$, and that the series

$$F(x) = \sum_{n=1}^{\infty} f_n(x)$$

converges for every $x \in [a, b]$. Prove that $F$ is continuous on $[a, b]$.

2. Let $g : \mathbb{R} \to \mathbb{R}$ be a $C^2$ function. Assume that $g$ and $g''$ are both bounded on $\mathbb{R}$. Show that $g'$ is bounded on $\mathbb{R}$

Hint: Show that there exists a sequence $(b_n)$, $n \in \mathbb{Z}$ such that $n < b_n \leq n + 1$ and $(g'(b_n))$ is a bounded sequence.

3. Define

$$u_n = \frac{1 \cdot 3 \cdot 5 \cdots (2n - 1)}{2 \cdot 4 \cdots 2n}$$

(a) Determine the radius of convergence $R$ of the series $\sum_{n \geq 1} u_n x^n$

(b) Study the convergence at $x = -R$ (hint: study $-\log(u_n)$).

(c) Study the convergence at $x = R$.

4. Let $(a_n)$ be a non negative sequence. Show that

$$\sum_{n \geq 1} a_n \text{ converges} \Rightarrow \sum_{n \geq 1} \sqrt{a_n} \text{ converges.}$$

Is the converse true?

5. Compute

$$\lim_{n \to \infty} \int_0^n \left( 1 + \frac{x}{2n} \right)^n e^{-x} dx.$$

6. Let $E$ be a measurable set of finite measure. For each $x \in \mathbb{R}$, let $f(x) = m(E \cap (\infty, x])$. Prove that $f$ is uniformly continuous on $\mathbb{R}$, and that $f(n) \to m(E)$ as $n \to \infty$. 
1. Consider the series
\[ f(x) = \sum_{n=1}^{\infty} \frac{nx}{1 + nx^2}. \]
(a) Show that the series converges for every real number \(x\).
(b) Show that for any \(\delta > 0\) the series converges uniformly for \(|x| \geq \delta\).
(c) Show that the function \(f\) defined above for all real numbers \(x\) is continuous at every non-zero \(x\). Is it continuous at \(x = 0\)? Justify your answer.

2. Let \(f\) be a real-valued differentiable function defined on \((-r, r)\), where \(r\) is a positive number. Show that \(f\) is an even function if and only if its derivative \(f'\) is an odd function.

3. Let \(f : \mathbb{R} \to \mathbb{R}\) be a continuous function satisfying the equation \(f(x + y) = f(x)f(y)\) for all real numbers \(x\) and \(y\).
   (a) Show that \(f(0) = 0\) or \(f(0) = 1\).
   (b) Show that if \(f(0) = 1\), then
   \[ f\left(\frac{m}{n}\right) = (f(1))^\frac{m}{n} \]
   for all integers \(m\) and \(n\), where \(n\) is non-zero.
   (c) Show that \(f\) is either the zero function or there exists a positive number \(a\) such that \(f(x) = a^x\) for all real numbers \(x\).

4. Let \(f : [0, 1] \to \mathbb{R}\) be differentiable function satisfying the conditions: \(f(0) = 0, \ |f'(x)| \leq |f(x)|\) for all \(0 < x < 1\). Prove that \(f\) is the zero function.

5. Let \(f\) be a real-valued continuous function defined on \([0, 1]\). Show that
\[ \lim_{n \to \infty} \int_{0}^{1} f(x^n) \, dx = f(0). \]

6. Let \(A\) be a Lebesgue measurable subset of \(\mathbb{R}\) with \(m(A) > 0\), where \(m\) denotes the Lebesgue measure. Show that for any \(0 < \delta < m(A)\) there exists a measurable subset \(B\) of \(A\) such that \(m(B) = \delta\). Hint: consider the function \(f(x) = m(A \cap [-x, x])\) for all \(x > 0\).
1. Prove that the function \( f(x) = \sqrt{x} \) is uniformly continuous on \([0, \infty)\).

2. Let \( a < b \). Suppose that the function \( f : [a, b] \to \mathbb{R} \) is bounded and Riemann integrable on \([c, b]\) for every \( a < c < b \). Prove that \( f \) is Riemann integrable on \([a, b]\) and \( \int_a^b f(x) \, dx = \lim_{c \to a} \int_c^b f(x) \, dx \).

3. (a) Is there a closed uncountable subset of \( \mathbb{R} \) which contains no rational numbers? Prove your assertion. (b) Is there an infinite compact subset of \( \mathbb{Q} \)? Prove your assertion. Here \( \mathbb{R} \) denotes the set of all real numbers equipped with the usual metric and \( \mathbb{Q} \) is the set of all rational numbers.

4. Consider the power series
\[
\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} x^n.
\]
(a) Prove that the series converges for \(|x| < 1\) and diverges for \(|x| > 1\). (b) Investigate the convergence and divergence of the series for \( x = \pm 1 \).

5. Let \((X, d)\) be a metric space and let \( A \) and \( B \) be two nonempty subsets of \( X \) such that \( A \cap B = \emptyset \). Prove that if \( A \) is closed and \( B \) is compact, then \( d(A, B) > 0 \), where \( d(A, B) \) denotes the distance between \( A \) and \( B \).

6. Let \( a_1 = 0 \) and for every positive integer \( n \geq 2 \), let
\[
a_n = \int_0^\infty \frac{x^n}{1 + x^2} \, dx.
\]
Show that the sequence \( \{a_n\} \) converges and find its limit.
ANALYSIS QUALIFYING EXAMINATION

January 10, 2011

Solutions to the problems are posted at [http://math.fau.edu/AnQualifiers/anqua_Jan2011sol.pdf](http://math.fau.edu/AnQualifiers/anqua_Jan2011sol.pdf)

1. Let $A, B$ be non-empty sets of real numbers such that $a \leq b$ for all $a \in A$, $b \in B$. Prove the following two statements are equivalent:

   (a) $\sup A = \inf B$.

   (b) For every $\epsilon > 0$ there exist $a \in A$ and $b \in B$ such that $b - a < \epsilon$.

2. Let $f : [0, \infty) \to \mathbb{R}$ be uniformly continuous and assume that

   $\lim_{b \to \infty} \int_{0}^{b} f(x) \, dx$

   exists and is finite. Prove that $\lim_{x \to \infty} f(x) = 0$.

3. Let $\psi : \mathbb{R} \to \mathbb{R}$ be defined by

   $\psi(x) = \begin{cases} 
   0 & \text{if } x < 0, \\
   x & \text{if } 0 \leq x < 1, \\
   1 & \text{if } x \geq 1.
   \end{cases}$

   Consider the series $\sum_{k=1}^{\infty} \frac{\psi(kx)}{k(1 + kx^2)}$.

   (a) Prove the series converges for all $x \in \mathbb{R}$.

   (b) Let $f(x) = \sum_{k=1}^{\infty} \frac{\psi(kx)}{k(1 + kx^2)}$. Prove $\limsup_{x \to 0^+} f(x) > 0$.

   **Hint:** $f(x) \geq \sum_{k=1}^{n} \frac{\psi(kx)}{k(1 + kx^2)}$; estimate for $x = 1/n$.

   (c) Prove $f$ is continuous on $(-\infty, 0) \cup (0, \infty)$ but discontinuous at 0.

4. Let $\mathcal{A}$ consist of all functions from $[0, \pi]$ to $\mathbb{R}$ that are finite linear combinations of elements of the set $\{\sin(nx) : n \in \mathbb{N}\}$; that is, $f \in \mathcal{A}$ if and only if $f(x) = \sum_{k=1}^{n} a_k \sin(kx)$ for some $n \in \mathbb{N}$, $a_1, \ldots, a_n \in \mathbb{R}$.

   (a) Prove: If $f : [0, \pi] \to \mathbb{R}$ is continuous and satisfies $f(0) = f(\pi) = 0$, then $f$ can be approximated uniformly by a sequence in $\mathcal{A}$.

   **Hint:** Add the constant function 1 to $\mathcal{A}$; take it away again later on.

   (b) Prove: If $f : [0, \pi] \to \mathbb{R}$ is continuous and satisfies $\int_{0}^{\pi} f(x) \sin nx \, dx = 0$ for all $n \in \mathbb{N}$, then $f(x) = 0$ for all $x \in [0, \pi]$.

5. Let $f_n : \mathbb{R} \to \mathbb{R}$ be measurable for $n = 1, 2, \ldots$. Let $a_n = \int_{\mathbb{R}} |f_n|$ for $n = 1, 2, \ldots$ and assume that $\sum_{n=1}^{\infty} a_n < \infty$. Prove:

   The series $\sum_{n=1}^{\infty} f_n$ converges almost everywhere.

6. Compute, and justify your computation,

   $\lim_{n \to \infty} \int_{0}^{1} \frac{\cos(x^n)}{1 + x^n} \, dx$. 

INSTRUCTIONS:

• Write only on one side of each of the sheets you hand in. Anything written on the back of a sheet might be ignored.

• Write your name on each sheet.

• Write clearly. A completely solved problem is worth more than several problems left half done.

1. Let \( \{a_n\} \) be a sequence of real numbers and let \( S \) be the set of all limits of subsequences of \( \{a_n\} \); that is, \( x \in S \) if and only if \( x \in \mathbb{R} \) and there exists a sequence of positive integers \( \{n_k\} \) such that \( n_1 < n_2 < n_3 < \cdots \) and such that \( \lim_{k \to \infty} a_{n_k} = x \). Prove: \( S \) is a closed subset of \( \mathbb{R} \).

2. Let \( \sum_{n=1}^{\infty} a_n \) be a convergent series of positive terms. Prove there exists a sequence of real numbers \( \{c_n\} \) such that \( \lim_{n \to \infty} c_n = \infty \) and such that \( \sum_{n=1}^{\infty} c_n a_n \) converges.

3. Consider the series \( \sum_{k=1}^{\infty} \frac{x}{k(1+kx^2)} \)
   (a) Prove the series converges for all \( x \in \mathbb{R} \).
   (b) Let \( f(x) = \sum_{k=1}^{\infty} \frac{x}{k(1+kx^2)} \). Prove \( f \) is continuous at all \( x \neq 0 \).
   (c) Is \( f \) continuous at 0?
   **Hint:** Accept or prove (depending on how much time you have left)
   \[
   \sum_{k=1}^{\infty} \frac{1}{k(1+kx^2)} \leq \frac{1}{1+x^2} + \int_{1}^{\infty} \frac{dt}{t(1+t^2)}.
   \]
   But notice that the integral becomes infinite as \( x \to 0 \). As an additional hint \( \frac{1}{t(1+t^2)} = \left( \frac{1}{t} - \frac{x^2}{1+x^2t} \right) \). Do not separate prematurely!

4. Let \( f : \mathbb{R} \to \mathbb{R} \) be differentiable; assume that \( |f(x)| \leq 1, |f'(x)| \leq 1 \) for all \( x \in \mathbb{R} \) and that \( f(0) = 0 \). Let \( \{a_n\} \) be a sequence of non zero real numbers. Prove: The sequence of functions \( g_n(x) = \frac{1}{a_n} f(a_n x) \) has a subsequence converging to a continuous function.

5. Assume \( f : \mathbb{R} \to \mathbb{R} \) and assume that \( \{x \in \mathbb{R} : f(x) \geq r\} \) is measurable for each rational number \( r \). Prove that \( f \) is measurable.

6. Assume \( f \) is Lebesgue integrable over the interval \([0,1]\). Prove that
   \[
   \lim_{n \to \infty} \frac{1}{n} \int_{0}^{1} \log \left( 1 + e^{nf(x)} \right) dx = \int_{\{x \in [0,1] : f(x) > 0\}} f(x) dx.
   \]
   **Hint:** Prove that \( \log(1 + e^t) \leq \log 2 + \max(t,0) \) for all \( t \in \mathbb{R} \).

7. Let \( K \) be a compact subset of \( \mathbb{R} \) with Lebesgue measure \( m(K) = 1 \). Let \( K_0 = \bigcap \{A : A \text{ is a compact subset of } K \text{ and } m(A) = 1\} \).
   Prove \( m(K_0) = 1 \) and if \( A \) is any proper compact subset of \( K_0 \), then \( m(A) < 1 \).
   **Hint:** Prove: The intersection of two, hence of any finite number, of compact subsets of \( K \) of measure 1, is again a compact subset of measure 1. Use this to conclude that if \( V \) is open in \( \mathbb{R} \) and \( K_0 \subset V \) then \( V \) must contain some compact subset of \( K \) of measure 1. Then use regularity of the Lebesgue measure; that is, use the fact that the measure of any measurable set is the infimum of the measure of all open sets containing it.
Complete as many problems as possible.

1. Construct a subset of $[0, 1]$ which is compact, perfect, nowhere dense, and with positive Lebesgue measure. (Be sure to prove your set has these four properties.)

2. [i] Suppose $A, B$ are nonempty, disjoint, closed subsets of a metric space $X$. Show that the function $f : X \to [0, 1]$ defined by
   $$f(x) = \frac{\text{dist}(x, A)}{\text{dist}(x, A) + \text{dist}(x, B)}$$
   is continuous with $f(x) = 0$ for all $x \in A$, $f(x) = 1$ for all $x \in B$, and $0 < f(x) < 1$ for all $x \in X \setminus (A \cup B)$.

   [ii] Show that if $X$ is a connected metric space with at least two distinct points, then $X$ is uncountable.

3. [i] Let $\{a_n\}_{n \geq 1}$ be a sequence of real numbers. Prove the Root Test: if $\limsup_{n \to \infty} \sqrt[n]{|a_n|} < 1$, then $\sum a_n$ is absolutely convergent, and if $\limsup_{n \to \infty} \sqrt[n]{|a_n|} > 1$, then $\sum a_n$ is divergent.

   [ii] Let $c_n = (\text{the nth digit of the decimal expansion of } \pi) + 1$. Prove that the series $\sum c_n x^n$ has radius of convergence equal to 1.

4. Let $\{f_n\}$ be a sequence of continuously differentiable functions on $[0, 1]$ with $f_n(0) = f'_n(0)$ and $|f'_n(x)| \leq 1$ for all $x \in [0, 1]$ and $n \in \mathbb{N}$. Show that if $\lim_{n \to \infty} f_n(x) = f(x)$ for all $x \in [0, 1]$, then $f$ is continuous on $[0, 1]$. Must the sequence converge? Must there be a convergent subsequence?

5. Suppose that $f, g, h : [a, b] \to \mathbb{R}$ satisfy $f(x) \leq g(x) \leq h(x)$ for all $x \in [a, b]$ and $f(x_0) = h(x_0)$ for some $x_0 \in (a, b)$. Prove that if $f$ and $h$ are differentiable at $x_0$, then $g$ is differentiable at $x_0$ with $f'(x_0) = g'(x_0) = h'(x_0)$.

6. Compute
   $$\lim_{n \to \infty} \int_1^n \frac{n(\sqrt{x} - 1)}{x^3 \log(x)} \, dx$$

7. Suppose $f : [a, b] \to \mathbb{R}$ is Riemann integrable on every subinterval $[a + \epsilon, b]$ for $0 < \epsilon < b - a$. Show that if $f$ is Lebesgue integrable on $[a, b]$, then the (improper) Riemann integral exists on $[a, b]$ and is equal to the Lebesgue integral. Is the converse true?
1. Let \( u : [a, b + 1] \to \mathbb{R} \) be a continuous function. For fixed \( \tau \in [0, 1] \) define \( v_\tau : [a, b] \to \mathbb{R} \) by \( v_\tau(t) = u(t + \tau) \). Show that \( \{v_\tau \mid \tau \in [0, 1]\} \) is a compact subset of \( C([a, b], \mathbb{R}) \).

2. Suppose \( A \) is a compact subset of \( \mathbb{R}^n \) and \( f : A \to \mathbb{R} \) is continuous. Prove that for every \( \epsilon > 0 \) there exists \( M > 0 \) such that \( |f(x) - f(y)| \leq M \|x - y\| + \epsilon \).

3. Suppose \( U \) is an open subset of \( \mathbb{R}^n \) and \( f : U \to \mathbb{R}^n \) is a homeomorphism. Prove that if \( f \) is uniformly continuous, then \( U = \mathbb{R}^n \).

4. Suppose \( \{C_n\}_{n \geq 1} \) is a nested decreasing sequence of nonempty, compact, connected subsets of a metric space. Prove that \( \bigcap_{n \geq 1} C_n \) is nonempty, compact, and connected.

5. For \( p > 1 \) compute
\[
\lim_{n \to \infty} \int_0^1 \frac{x^p}{x^2 + (1 - nx)^2} \, dx.
\]

6. Let \( f : \mathbb{R} \to \mathbb{R} \) be a Lebesgue integrable function, \( E \subset \mathbb{R} \) a measurable set, and \( \omega > 0 \). Define \( \omega E = \{\omega x \mid x \in E\} \).

   [a] Show that \( m(\omega E) = \omega m(E) \).

   [b] Show that the function \( g : E \to \mathbb{R} \) defined by \( g(x) = f(\omega x) \) is Lebesgue integrable and
\[
\int_E f(\omega x) \, dx = \frac{1}{\omega} \int_{\omega E} f(x) \, dx.
\]

7. Prove that if \( f : [0, \infty) \to \mathbb{R} \) is Lebesgue integrable, then
\[
\lim_{n \to \infty} \frac{1}{n} m \left( \left\{ x \geq n \mid |f(x)| \geq \frac{1}{n} \right\} \right) = 0.
\]
Name (Please print)__________________________________________________________

1. Assume that \( \{a_n\} \) is a monotone decreasing sequence with \( a_n \geq 0 \). If \( \sum_{n=1}^{\infty} a_n < \infty \), show that \( \lim_{n \to \infty} na_n = 0 \). Is the converse true?

2. Let \( f : \mathbb{R} \to \mathbb{R} \) be defined by

\[
f(x) = \begin{cases} 
  x & \text{if } x \text{ is irrational} \\
  p \sin \frac{1}{q} & \text{if } x = \frac{p}{q} \text{ in lowest terms } (q > 0) 
\end{cases}
\]

At what points is \( f \) continuous?

3. Let \( f(x) = (x^2 - 1)^n \), and \( g = f^{(n)} \) (i.e., the \( n \)th derivative of \( f \)). Show that the polynomial \( g \) has \( n \) distinct real roots, all in the interval \([-1, 1]\).

4. Let \( X \) be a nonempty set, and for any two functions \( f, g \in \mathbb{R}^X \) (the set of all functions from \( X \) to \( \mathbb{R} \)) let

\[
d(f, g) = \sup_{x \in X} \frac{|f(x) - g(x)|}{1 + |f(x) - g(x)|}.
\]

Establish the following:

(a) \( (\mathbb{R}^X, d) \) is a metric space.

(b) A sequence \( \{f_n\} \subseteq \mathbb{R}^X \) satisfies \( d(f_n, f) \to 0 \) for some \( f \in \mathbb{R}^X \) if and only if \( \{f_n\} \) converges uniformly to \( f \).

5. Let \( E = \{(x, y) \in \mathbb{R}^2 : 9x^2 + y^4 = 1\} \). Show that \( E \) is compact and connected.

6. If \( \int_A f = 0 \) for every measurable subset \( A \) of a measurable set \( E \), show that \( f = 0 \) a.e. in \( E \).

7. Evaluate

\[
\lim_{n \to \infty} \int_{[0,1]} \left( 1 - e^{-x^2/n} \right) x^{-1/2} dx.
\]
1. If \( \{x_n\} \) is a convergent sequence in a metric space, show that any rearrangement of \( \{x_n\} \) converges to the same limit.

2. Consider the series
   \[
   \sum_{n=1}^{\infty} \frac{1}{1 + x^n}.
   \]
   (a) Show that the series diverges for \( |x| < 1 \), and converges for \( |x| > 1 \).
   (b) Let \( f(x) = \sum_{n=1}^{\infty} \frac{1}{1 + x^n} \). Find the set where \( f \) is continuous.

3. Let \( G \) be a non-trivial additive subgroup of \( \mathbb{R} \). Let
   \[
   a = \inf \{ x \in G : x > 0 \}.
   \]
   Prove: If \( a > 0 \) then \( G = \{ na : n \in \mathbb{Z} \} \), otherwise (i.e., if \( a = 0 \)) \( G \) is dense in \( \mathbb{R} \).

4. Consider the function \( f : [-1, 1] \rightarrow \mathbb{R} \) defined by
   \[
   f(x) = \begin{cases} 
   \frac{x}{2} + x^2 \sin \left( \frac{1}{x} \right), & \text{if } x \neq 0; \\
   0, & \text{if } x = 0.
   \end{cases}
   \]
   Prove that \( f'(0) \) is positive, but that \( f \) is not increasing in any open interval that contains \( 0 \).

5. For \( x \in [-1, 1] \) and \( n \in \mathbb{N} \), define
   \[
   f_n(x) = \frac{x^{2n}}{1 + x^{2n}}.
   \]
   (a) Find a function \( f_0 \) on \( [-1, 1] \) such that \( \{f_n\} \) converges pointwise to \( f_0 \).
   (b) Determine whether \( \{f_n\} \) converges uniformly to \( f_0 \).
   (c) Calculate \( \int_{-1}^{1} f_0(x)dx \) and determine whether
   \[
   \lim_{n \to \infty} \int_{-1}^{1} f_n(x)dx = \int_{-1}^{1} f_0(x)dx.
   \]

6. Let \( (X, \mathcal{A}, \mu) \) be a measurable space and \( \{f_n\} \) a sequence of measurable functions. We say that \( \{f_n\} \) converges in measure to \( f \) if for every \( \varepsilon > 0 \),
   \[
   \lim_{n \to \infty} \mu \left( \{ x \in X : |f_n(x) - f(x)| > \varepsilon \} \right) = 0.
   \]
   Show that if \( \mu \) is a finite measure, and \( f_n \rightarrow f \) a.e., then \( f_n \rightarrow f \) in measure. Give an example of a sequence which converges in measure, but does not converge a.e.

7. Show that
   \[
   \lim_{n \to \infty} \int_{0}^{n} \left( 1 + \frac{x}{n} \right)^n e^{-2x} dx = 1.
   \]
Analysis Qualifying Examination Spring 2008

Complete as many problems as possible.

1. Let \( f(x) = \sin \left( \frac{1}{x} \right) \) for \( x > 0 \). Prove that \( f \) is uniformly continuous on \((a, \infty)\) for every \( a > 0 \). Is \( f \) uniformly continuous on \((0, \infty)\)? Justify your answer.

2. Let \( f \) be continuous on \([0, 1]\) and differentiable in \((0, 1)\) such that \( f(0) = f(1) \). Prove that there exists \( 0 < c < 1 \) such that \( f'(1 - c) = -f'(c) \).

3. Suppose \( X, Y \) are metric spaces, \( X \) is compact, and \( f : X \rightarrow Y \) is a continuous bijection. Prove that \( f \) is a homeomorphism.

4. Let \( C([0, 1], \mathbb{R}) = \{ f \mid [0, 1] \rightarrow \mathbb{R} \mid f \) is continuous\} be the metric space of continuous functions on \([0, 1]\) with the metric \( d(f, g) = \| f - g \|_\infty = \sup_{x \in [0,1]} |f(x) - g(x)| \). Show that the unit ball \( \{ f \in C([0, 1], \mathbb{R}) \mid \| f \|_\infty \leq 1 \} \) is not compact.

5. Let \( f : [0, 1] \rightarrow \mathbb{R} \) be continuously differentiable on \([0, 1]\) with \( f(0) = 0 \). Prove that

\[
\sup_{0 \leq x \leq 1} |f(x)| \leq \int_0^1 |f'(x)| \, dx \leq \left( \int_0^1 |f'(x)|^2 \, dx \right)^{\frac{1}{2}}
\]

6. Compute

\[
\lim_{n \to \infty} \int_0^1 \sqrt{1 + x^n \sin(nx)} \, dx.
\]

Justify your answer.

7. Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) be Lebesgue integrable on \( \mathbb{R} \). Prove that

\[
\lim_{h \to 0} \int_{\mathbb{R}} |f(x + h) - f(x)| \, dx = 0.
\]
In some cases partial credit may be given, but you should endeavor to fully complete as many problems as possible.

1. Give an example of a sequence of Riemann integrable functions \( f_n : [0, 1] \to \mathbb{R} \) for which \( f_n \to 0 \) pointwise on \([0, 1]\) but
   \[
   \lim_{n \to \infty} \int_{0}^{1} f_n(x) \, dx \neq 0.
   \]

2. Let \( k \) be a fixed positive integer, and let \( A \) be the set of all polynomials of the form
   \[
p(x) = a_k x^k + a_{k+1} x^{k+1} + \cdots + a_n x^n, \text{ where } n \in \mathbb{N}, n \geq k, \text{ and } a_i \in \mathbb{R}.
   \]
   Prove that \( A \) is dense in \( C[a, 1] \) for every \( 0 < a < 1 \). Is it also dense in \( C[0, 1] \)? Prove your conclusion.

3. (a) Suppose that \( f : [a, b] \to \mathbb{R} \) is differentiable, and that \( f' \) is bounded on \([a, b]\). Prove that \( f \) is of bounded variation on \([a, b]\). (b) Define \( f(x) = x^2 \cos(1/x) \) for \( 0 < x \leq 1 \) and \( f(0) = 0 \). Using (a), prove that \( f \) is of bounded variation on \([0, 1]\).

4. Let \( U \) be an open subset of \( \mathbb{R}^n \) and let \( f : U \to \mathbb{R}^m \). (a) State the definition that \( f \) is differentiable at \( p \in U \). (b) Show that every linear transformation \( T : \mathbb{R}^n \to \mathbb{R}^m \) is differentiable at every \( p \in \mathbb{R}^n \). What is the derivative of \( T \) at \( p \)?

5. Define \( f : [0, 1] \to \mathbb{R} \) by
   \[
f(x) = \begin{cases} 
0, & \text{if } x \text{ is rational} \\
x^2, & \text{if } x \text{ is irrational}
\end{cases}
   \]
   Prove that \( f \) is not Riemann integrable on \([0, 1]\) but it is Lebesgue integrable on \([0, 1]\). Find the Lebesgue integral of \( f \).

6. Let \( X \) be a compact metric space and let \( \{f_n\} \) be a sequence of isometries on \( X \). Prove that there exists a subsequence \( \{f_{n_k}\} \) that pointwise converges to an isometry \( f \) on \( X \). Recall that \( f : X \to X \) is called an isometry if \( d(f(x), f(y)) = d(x, y) \) for all \( x, y \in X \), where \( d \) is the metric on \( X \).
Analysis Qualifying Examination

Spring 2007

Name: ________________________ Last Four Digits of Your Student Number: ____________

Full completion of at least 4 problems will result in a pass. In some cases partial credit may be given, but you should endeavor to fully complete as many problems as possible.

1. Define the sequence \( \{a_n\} \) by setting
   \[
   a_n = \frac{n!}{n^n}
   \]
   for all positive integers \( n \).
   
   (a) Prove that the sequence \( \{a_n\} \) is strictly decreasing and bounded.
   
   (b) Find the limit of the sequence \( \{a_n\} \).
   
   (c) Find the limit of the sequence \( \left\{ \frac{a_{n+1}}{a_n} \right\} \).

2. Let \( A \) and \( B \) be two nonempty open subsets of \( \mathbb{R} \), the set of real numbers equipped with the usual metric. Suppose that \( x < y \) for all \( x \in A \) and all \( y \in B \). Prove that there exists a real number \( z \) such that \( x < z < y \) for all \( x \in A \) and all \( y \in B \).

3. Consider the series
   \[
   \sum_{n=1}^{\infty} \frac{(-1)^n}{n^x},
   \]
   where \( x \) is a real variable. Find the values of \( x \) for which the series converges and find the values of \( x \) for which the series converges absolutely. Show that the function
   \[
   f(x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^x}
   \]
   defined for \( x > 0 \) is continuous on the interval \((0, +\infty)\).

4. Suppose that \( f : (0, +\infty) \to \mathbb{R} \) is differentiable and that there exists a constant \( \alpha \in \mathbb{R} \) such that \( xf'(x) = \alpha f(x) \) for all \( x > 0 \) and \( f(1) = 1 \). Prove that \( f(x) = x^\alpha \) for all \( x > 0 \).

5. Let \( K \) be a compact subset of \( \mathbb{R}^n \) and let \( f : K \to \mathbb{R} \) be a continuous function. Define
   \[
   M = \{ x \in K : f(x) \text{ is the maximum value of } f \text{ on } K \}.
   \]
   Prove that \( M \) is a compact subset of \( \mathbb{R}^n \).

6. Let \( f : \mathbb{R}^n \to \mathbb{R}^m \) be a function such that \( \|f(x) - f(y)\| \leq \|x - y\|^2 \) for all \( x, y \in \mathbb{R}^n \), where \( \| \cdot \| \) denotes the Euclidean norm. Prove that \( f \) is a constant function.
Full completion of at least 4 problems will result in a pass. In some cases partial credit may be given, but you should endeavor to fully complete as many problems as possible.

1. Let $A$ be a bounded subset of $\mathbb{R}$. Define $D = \{x - y : x \in A$ and $y \in A\}$. Prove that $D$ is bounded and $\sup(D) = \sup(A) - \inf(A)$. State and prove a similar result for $\inf(D)$.

2. Prove that the series
   \[\sum_{n=1}^{\infty} \left(\frac{x^n}{n} - \frac{x^{n+1}}{n+1}\right)\]
   converges uniformly on the interval $[-1, 1]$.

3. Let $f : (0, 1) \to \mathbb{R}$ be differentiable such that $f'$ is bounded on $(0, 1)$. Prove that both $\lim_{x \to 0^+} f(x)$ and $\lim_{x \to 1^-} f(x)$ exist.

4. Let $f(0,0) = 0$ and
   \[f(x,y) = \frac{2xy(x^2 - y^2)}{x^2 + y^2}\]
   if $(x,y) \neq (0,0)$. Prove that $f$ is continuous everywhere and the two partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist everywhere. Is $f$ differentiable at $(0,0)$? Justify your conclusion.

5. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous and satisfies the equation
   \[f(x) = 1 + \int_{0}^{x} f(t) \, dt\]
   for all real numbers $x$. Prove that $f(x) = e^x$.

6. Let $f : [a, b] \to \mathbb{R}$ be continuous, where $[a, b]$ is a bounded closed interval. Define
   \[G = \{(x, f(x)) : x \in [a, b]\}.
   
   $G$ is called the graph of $f$. Prove that $G$ is a connected subset of $\mathbb{R}^2$, where $\mathbb{R}^2$ is equipped with the standard metric.
Analysis Qualifying Examination
Spring 2006

Name: ___________________ Last Four Digits of Your Student Number: ____________

Full completion of at least 4 problems will result in a pass. In some cases partial credit may be given, but you should endeavor to fully complete as many problems as possible.

1. (a) Show that $|\sin x - \sin y| \leq |x - y|$ for all real numbers $x$ and $y$.
   
   (b) Show that there exists no positive constant $0 < c < 1$ such that $|\sin x - \sin y| \leq c|x - y|$ for all real numbers $x$ and $y$.

2. Define a sequence $\{x_n\}$ as follows:

   $$x_n = \frac{1}{1 \cdot 4} + \frac{1}{2 \cdot 5} + \frac{1}{3 \cdot 6} + \cdots + \frac{1}{n \cdot (n + 3)}.$$

   Prove that $\{x_n\}$ converges and find its limit.

3. Assume that $f$ is differentiable at $a$ and that $\{x_n\}$ and $\{z_n\}$ are two sequences that converge to $a$ and that $x_n < a < z_n$ for all $n$. Prove that

   $$\lim_{n \to \infty} \frac{f(x_n) - f(z_n)}{x_n - z_n} = f'(a).$$

4. Assume that $f$ is a continuous real-valued function on $\mathbb{R}$ and that

   $$\lim_{x \to +\infty} f(x) = \lim_{x \to -\infty} f(x) = 0.$$

   Prove that if $f$ does not have a maximum then it has a minimum and if it does not have a minimum then it has a maximum.

5. Prove that the series

   $$\sum_{n=1}^{\infty} (-1)^n \frac{x^2 + n}{n^2}$$

   converges uniformly in every bounded interval, but does not converge absolutely for any real value of $x$.

6. Suppose that $(M, d)$ is a metric space and that $\{x_n\}$ is a Cauchy sequence in $(M, d)$. Suppose also that $\{x_n\}$ has a convergent subsequence. Prove that $\{x_n\}$ converges.
1. For every positive integer \( n \), define
\[
a_n = \int_0^1 \frac{x^n}{1 + x^n} \, dx.
\]
Prove that \( 0 \leq a_{n+1} \leq a_n \) for all \( n \). Is the sequence \( \{a_n\} \) convergent? If so, find its limit.

2. Let \( f : I \to \mathbb{R} \) be continuous, where \( I = (a, b) \) is an open interval in \( \mathbb{R} \). Assume that \( c \in I \) and \( f \) is differentiable in \( (a, c) \) and in \( (c, b) \), and assume that \( L = \lim_{x \to c} f'(x) \) exists (and is finite). Prove that \( f \) is differentiable at \( c \) and \( f'(c) = L \).

3. Consider the series
\[
\sum_{n=0}^{\infty} (e^{-nx} - e^{-2nx}).
\]
(a) Prove that it converges for all \( x \geq 0 \).
(b) For \( x \geq 0 \), set
\[
f(x) = \sum_{n=0}^{\infty} (e^{-nx} - e^{-2nx}).
\]
Prove that \( f \) is continuous at all \( x > 0 \) but discontinuous at \( x = 0 \).

4. Let \( f \) and \( g \) be two real valued differentiable functions defined on \( \mathbb{R} \). Suppose that for all \( x \in \mathbb{R} \)
\[
\begin{align*}
f^2(x) + g^2(x) &= 1 \\
f'(x) &= g(x) \quad \text{and} \quad g'(x) = -f(x) \\
f(0) &= 0 \quad \text{and} \quad g(0) = 1.
\end{align*}
\]
Prove that \( f \) is the sine function and \( g \) is the cosine function.

*Hint:* Consider the function \( F(x) = (f(x) - \sin x)^2 + (g(x) - \cos x)^2 \).

5. Let \( C \) be a connected subset of \( \mathbb{R} \). Prove that if \( f : C \to \mathbb{R} \) is continuous and \( f(x) \) is an integer for all \( x \in C \), then \( f \) is a constant function.

6. Let \( U \) be an open subset of \( \mathbb{R}^n \) and let \( f : U \to U \) be one-to-one and onto and differentiable in \( U \). Let \( f^{-1} \) denote the inverse of \( f \). Prove that if \( f'(c) = 0 \) for some \( c \in U \), then \( f^{-1} \) is not differentiable in \( U \), where \( f'(c) \) denotes the differential of \( f \) at \( c \); another notation is \( df(c) \).
Analysis Qualifying Examination

You should try all the six problems. But please tell us which five of them you want us to grade.

1. If $f$ is continuous on $[a, b]$, differentiable in $(a, b)$, and $f'(x) = 0$ for all $x \in (a, b)$, prove that $f$ is a constant on $[a, b]$.

2. Let $f$ be a function defined on $[0, +\infty)$ such that (i) $f$ is continuous on $[0, +\infty)$, (ii) $f$ is differentiable in $(0, +\infty)$, (iii) $f'(x) \geq 1$ for all $x > 0$, and (iv) $f(0) = 0$. Fix $x_0 > 0$ and define $x_n = f(x_{n-1})$ for all $n \geq 1$.

   (a) Prove that $x_{n+1} \geq x_n$ for all $n$.

   (b) Prove that if the sequence $\{x_n\}$ converges then $\{x_n\}$ is a constant sequence, that is, $x_n = x_1$ for all $n \geq 1$.

3. Given the power series

$$
\sum_{n=1}^{\infty} \frac{2^n n!}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)} x^n.
$$

Find all $x$ for which the series converges.

4. Let $f$ be a decreasing function defined on $[1, +\infty)$. Suppose that $f(x) > 0$ for all $x \geq 1$. Prove that the sequence $\{D_n\}$ defined by

$$
D_n = f(1) + f(2) + \cdots + f(n) - \int_{1}^{n} f(t) \, dt
$$

is non-negative and decreasing and hence converges.

5. Let $\mathbb{R}$ be the set of real numbers equipped with the usual metric. Prove that every uncountable subset of $\mathbb{R}$ has at least one limit point.

6. Let $(X, d)$ be a compact metric space. Let $f : X \to X$ be a function such that $d(f(x), f(y)) < d(x, y)$ for all $x, y \in X$ with $x \neq y$. (a) Prove that the function $F$ defined by $F(x) = d(f(x), x)$ for $x \in X$ is continuous on $X$. (b) Using (a) prove that $f$ has a unique fixed point, that is, there exists a unique $z \in X$ such that $f(z) = z$. 
Solve as many problems as you can. You do not have to solve all to pass the qualifying exam.

1. Let $f$ be defined for all real $x$, and suppose that

$$|f(x) - f(y)| \leq (x - y)^2$$

for all real $x$ and $y$. Prove that $f$ is differentiable and $f'(x) = 0$ for all $x$. What can you say about the function $f$?

2. If $0 < x < \frac{\pi}{2}$, prove that

$$\frac{2}{\pi} < \frac{\sin x}{x} < 1.$$

3. (a) Let $f$ be a continuous function on $[a, b]$ with $a < b$. Suppose that $f(x) \geq 0$ for all $x \in [a, b]$ and $\int_{a}^{b} f(x) \, dx = 0$. Prove that $f(x) = 0$ for all $x \in [a, b]$.

(b) Let $g$ be a real continuous function on $[0, 1]$ such that

$$\int_{0}^{1} g^2(x) \, dx = \frac{1}{3} \quad \text{and} \quad \int_{0}^{1} xg(x) \, dx = \frac{1}{3}.$$

Using (a), prove that $g(x) = x$ for all $x \in [0, 1]$.

4. Given the power series

$$\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}.$$

Find all real $x$ for which the series converges.

5. Let $X$ be a nonempty set. For all $x, y \in X$, define $d(x, y) = 1$ if $x \neq y$ and $d(x, y) = 0$ if $x = y$. Prove that $d$ is a metric on $X$. Which subsets of the resulting metric space $(X, d)$ are open? Which are closed? Which are compact? Prove your statements.

6. Define $f(0, 0) = 0$ and

$$f(x, y) = \frac{x^3}{x^2 + y^2} \quad \text{if} \quad (x, y) \neq (0, 0).$$

(a) Prove that $f$ is continuous at $(0, 0)$.

(b) Prove that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist for all $x$ and $y$. 
