Well-Founded Iterations of Infinite Time Turing Machines

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Iteration and hyper-iteration/feedback

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- Turing jump $\mapsto$ hyperarithmetic sets
- Inductive definitions $\mapsto$ the $\mu-$calculus
- ITTMs $\mapsto$???

First Definitions

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- the machine is in a dedicated state
- the head is on the 0^{th} cell
- the content of a cell is limsup of the previous contents (i.e. 0 if eventually 0, 1 if eventually 1, 1 if cofinally alternating)

Writable reals and ordinals

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Proposition

$R \subseteq \omega$ is writable iff $R \in L_\lambda$.

Eventually writable reals and ordinals

Definition

$R \subseteq \omega$ is **eventually writable**
if its characteristic function is on the output tape, never to change, of a computation.

An ordinal α is **eventually writable**
if some real coding α (via some standard representation) is eventually writable.

$\zeta :=$ sup $\{\alpha \mid \alpha$ is eventually writable$\}$

Proposition

$R \subseteq \omega$ is eventually writable iff $R \in L_\zeta$.

Accidentally writable reals and ordinals

Definition

$R \subseteq \omega$ is **accidentally writable**
if its characteristic function is on the output tape at any time during a computation.

An ordinal α is **accidentally writable**
if some real coding α (via some standard representation) is accidentally writable.

$\Sigma :=$ sup $\{\alpha \mid \alpha$ is accidentally writable$\}$

Proposition

$R \subseteq \omega$ is accidentally writable iff $R \in L_\Sigma$.

Summary and conclusions

λ is the supremum of the writables.
ζ is the supremum of the eventually writables.
Σ is the supremum of the accidentally writables.
Clearly, $\lambda \leq \zeta \leq \Sigma$.

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Theorem

*(Welch) ζ is the least ordinal α such that L_α has a Σ_2-elementary extension. (ζ is the least Σ_2-**extendible** ordinal.) The ordinal of that extension is Σ. L_λ is the least Σ_1-elementary substructure of L_ζ.*

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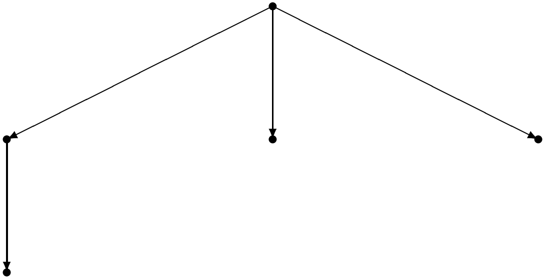
Proposition

The definitions of λ, ζ, and Σ relativize (to $\lambda^\blacktriangledown, \zeta^\blacktriangledown$, and $\Sigma^\blacktriangledown$) to computations from $0^\blacktriangledown$. Furthermore, $\zeta^\blacktriangledown$ is the least Σ_2-extendible limit of Σ_2-extendibles, the ordinal of its Σ_2 extension is $\Sigma^\blacktriangledown$, and $\lambda^\blacktriangledown$ is the ordinal of its least Σ_1-elementary substructure.

Time to iterate

ITTMs with arbitrary iteration:
A computation may ask a convergence question about another computation. This can be considered calling a sub-computation. That sub-computation might do the same. This can continue, generating a *tree of sub-computations*. Eventually, perhaps, a computation is run which calls no sub-computation. This either converges or diverges. That answer is returned to its calling computation, which then continues.

Good examples



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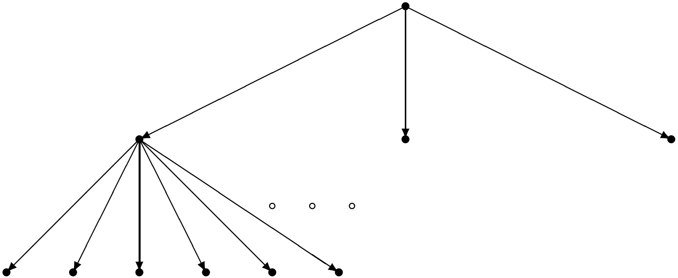

Bad example

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One can naturally define the course of a computation if and only if the tree of sub-computations is well-founded. How is this to be dealt with?

Option I

When the main computation makes a sub-call, the call must be made with an ordinal. When a sub-call makes a sub-call itself, that must be done with a smaller ordinal. The definitions of λ, ζ, and Σ relativize (to $\lambda^{it\blacktriangledown}, \zeta^{it\blacktriangledown}$, and $\Sigma^{it\blacktriangledown}$).

Results

Definition

β is 0- (or 1-) extendible if its Σ_2-extendible.
β is $(\alpha+1)$-extendible if its a Σ_2-extendible limit of α-extendibles.
β is κ-extendible if its a Σ_2-extendible limit of α-extendibles for each $\alpha < \kappa$.

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Proposition

$\zeta^{it\blacktriangledown}$ is the least κ which is κ-extendible, $\Sigma^{it\blacktriangledown}$ is its Σ_2 extension, and $\lambda^{it\blacktriangledown}$ its least Σ_1 substructure.

Option II

Allow all possible sub-computation calls, even if the tree of sub-computations is ill-founded, and consider only those for which the tree of sub-computations just so happens to be well-founded.

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Allow all possible sub-computation calls, even if the tree of sub-computations is ill-founded, and consider only those for which the tree of sub-computations just so happens to be well-founded. So some legal computations have an undefined result. Still, among those with a defined result, some computations are halting, and some divergent computations have a stable output.

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Proof.

Given e, run the computation of ϕ_e. Keep asking "if I continue running this computation until cell 0 changes, is that computation convergent or divergent?" Eventually you will get "divergent" as your answer. Then go on to cell 1, then cell 2, etc. After going through all the natural numbers, you know the real on your output tape is the eventually writable real you want. So halt. $\square$

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SECOND BIG FACT
If a real is accidentally writable in this fashion, then it's writable.

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QUESTION
Why isn't this a contradiction? Why can't you diagonalize?

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ANSWER

You can't run a universal machine.

As soon as a machine with code for a universal machine makes an ill-founded sub-computation call, it freezes.

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Claim In order to understand the writable reals in this context, one needs to understand the freezingly writable reals. One also needs to understand the tree of sub-computations for freezing computations.

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Proposition

Options a) and b) are incompatible: there cannot be one tree of sub-computations with more than Λ-much splitting beneath a node and another with the splittings beneath all the nodes cofinal in Λ.

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to be continued ...

References

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