Ax-Grothendieck and the Interplay Between Model Theory and Algebra

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The goal of this talk is to prove the Ax-Grothendieck theorem, namely that if $f : \mathbb{C}^n \to \mathbb{C}^n$ is a polynomial function which is injective, then it is surjective. The result is interesting, but its proof is even more so.

The proof uses a branch of mathematical logic called *model theory*, which is the roughly the study of the relationship between axiomatic systems and structures in which the axioms hold. For instance, a model theorist would study a given field by analyzing its relationship to the field axioms, rather than relationships between it and other fields (like homomorphisms). As an example of this kind of analysis, Ax-Grothendieck is proven over finite fields and their algebraic closures. Then, the novel techniques offered by model theory are employed to "lift" the result up to $\mathbb{C}$. These powerful model-theoretic tools allow us to prove relationships between (algebraic closures of) finite fields and $\mathbb{C}$, despite there not even being a homomorphism between them.

The speaker will use Ax-Grothendieck to highlight just one of the ways in which model theory (and foundations as a whole) appears in algebra, highlighting a bridge between two areas traditionally seen as being very far apart.