A lattice-theoretic view of some special ideals of subrings of commutative rings

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Abstract

If A is a commutative ring with identity and B is a subring of A, the pair $B \subseteq A$ is said to satisfy the Lying Over property (abbreviated "LO property") if $\mathrm{Spec}(B) = \{B \cap P \mid P \in \mathrm{Spec}(A)\}$. This concept was introduced by Kaplansky. For such a pair, we study the relationship between the radical ideals of the subring and the contractions of radical ideals of the bigger ring. This we do in the category **AFrm** of algebraic frames. We show that the pair $B \subseteq A$ satisfies the LO property if and only if the induced morphism $\mathrm{RId}(B) \to \mathrm{RId}(A)$ is a monomorphism in this category. A stronger property is one that requires over and above the LO property that $\mathrm{Max}(B) = \{B \cap M \mid M \in \mathrm{Max}(A)\}$. We call it the Strong Lying Over property (SLO property). As shown by Rudd, it is satisfied by any pair $I + \mathbb{R} \subseteq C(X)$, where I is an ideal of $C(X)$. We show, among other things, that in a class of rings properly containing all the rings $C(X)$, if $B \subseteq A$ satisfies the SLO property, then the z-ideals of the smaller ring are precisely the contractions to it of the z-ideals of the bigger ring.