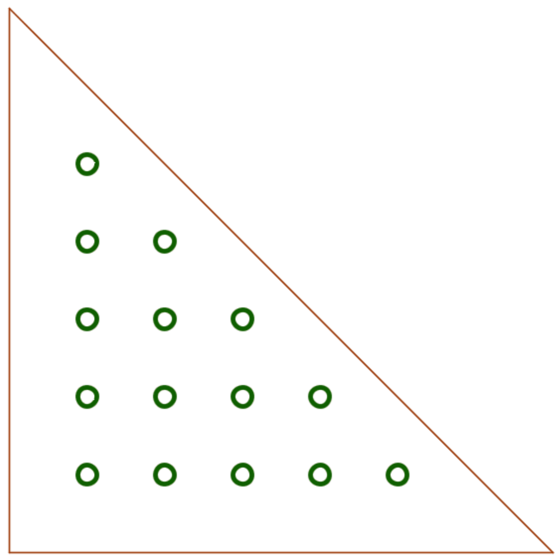
FAU Math Circle

4/27/2024

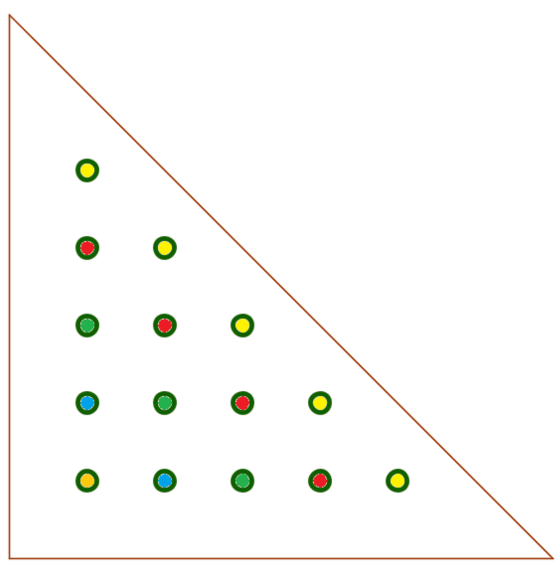
Solutions to the problems

Pegging Along Five (5) yellow pegs, four (4) red pegs, three (3) green pegs, two (2) blue pegs and one (1) orange peg are to be placed on a triangular peg board, like the one pictured below. In how many different ways can this be done if no horizontal row or vertical column is to contain two pegs of the same color.



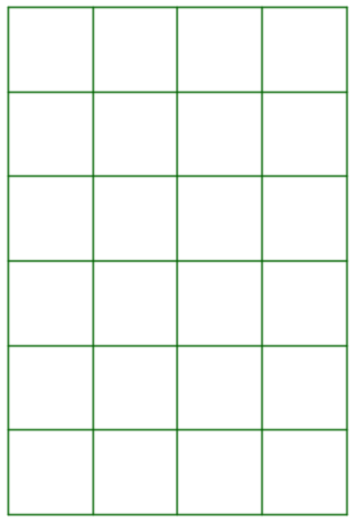
Same question if we relax the conditions, allow the same color in some columns, but no horizontal row is to contain two pegs of the same color.

Solution. If we don't place a yellow peg in the top hole we will be forced to have either two yellow pegs in the same (lower) row, or two in the same column. Reasoning this way it is easy to see that there is only one way the pegs can be placed so no horizontal row or vertical column is to contain two pegs of the same color.

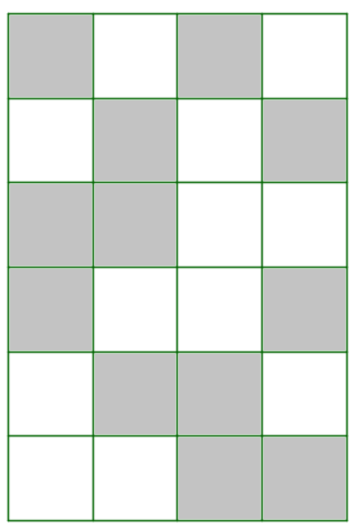


If we relax conditions as stated, we still have to put a yellow peg on top and there has to be on in each row. So this gives us $1 \times 2 \times 3 \times \times 4 \times 5 = 120$ different ways in which we can place the yellow peg. Since the top hole is occupied, to avoid having two red pegs in the same row, we have to place a red peg in the hole of the second row left free by the yellow peg, another one in one of the two holes left free in the third row by the yellow peg, and so forth; so for each possible way in which we can place the yellow pegs, we have $1 \times 2 \times 3 \times 4$ ways we can place the red pegs. Similarly, for each placement of the yellow and red pegs we will have $1 \times 2 \times 3 = 6$ ways of placing the green pegs, for each placement of the yellow, red, and green pegs we will have $1 \times 2 = 2$ ways in which we can place the blue pegs. Once all the yellow, red, green and blue pages are in, there is only one place for the orange peg. We thus have a total of $120 \times 24 \times 6 \times 2 \times 1 = \boxed{34560}$ different ways in which we can place the pegs.

Shading the Grid. One wishes to shade 12 squares of the grid below in such a way that every column has exactly three shaded squares and every row has exactly two shaded squares.



In how many different ways can one do this? Here is an example of a legal shading.



Solution. It helps to know a little bit about combinations; what we saw in the first and second sessions of our meetings..

Drawing some pictures can be very helpful. We will go column by column.

Column 1 We have "6 choose 3= 20" different ways in which we can shade three of the 6 squares of column 1. For each one of these shadings we have the following possibilities for the othrt columns.

a) In column 2 we shade the same rows as are shaded in column 1. Then there is no choice left for columns 3 and 4 but shade the remaining rows. **1 way.**

b) In column 2 we shade two rows in common with the ones shaded in column 1 (3 choices), and a fourth tow (3 choices). It leaves two so far rows with a single shade; they need to be shaded in columns 3 and 4, so one extra row is left to be shaded in columns 3, 4. There are two choices for the row in column three, none for column four. $3 \times 3 \times 2 =$ **18 ways**.

c) In column 2 we shade only one row in common with column 1 (3 choices), and shade a fourth and fifth row not shaded in column 1 (also 3 choices). The last row must be shaded in columns 3 and 4. We now have 4 rows left with only one square shaded; 2 will have to be shaded in column 3 ("4 choose 2 =6" ways), with column 4 being decided. $3 \times 3 \times 6 =$ **54 ways.**

d) Columns 1 and 2 have no rows shaded in common (1 way). Then we can shade any three squares of column 3 (20 ways) , and column 4 is decided. **20 ways**

Thus, for each one of the 20 ways to shade the squares of column 1, we have $1 + 18 + 54 + 20 = 93$ choices, there being 20 choices for column 1 this gives a total of $\boxed{20 \times 93 = 1860 \text{ ways}}$.

Math Clubs. Of the students attending a high school, 60% are girls and 45% of all the students are members of the Math Club. One day, the school gets 20 additional students, all of them boys and 19 of them join the math club. Now 55% of the students are girls (of course, the total number of girls has not changed, just their percentage). How many students are now members of the math club?

Solution. One way of doing this is finding out first the total number of students. Let's give that number a name; let s be the total number of students **before the 20 boys were added**. Let g be the total number of girls. We then know:

$$\begin{aligned} g &= 0.6s \\ g &= 0.55(s+20) = 0.55s + 11.2 \end{aligned}$$

Subtracting we get $0 = 0.05s - 11$; that is, $s = 11/0.05 = 220$. Now 45% of 220 is 89. Adding the 19 new boys who joined the Math Club we get $\boxed{118}$ members of the club.

Eleven Rules How many 5 digit integers n have the property that their remainder r and their quotient q when divided by 100, add up to a multiple of 11. So you want to know how many 5 digit integers n have the following property: When divided by 100 their quotient is q, the remainder is r and $r + q$ is a multiple of 11.

Solution. If $n = abcde$, where a, b, c, d, e are digits, that means that $n = 10^4a + 10^3b + 10^2c + 10d + e = 100(10^2a + 10b + c) + 10d + e$ so that $r = 10d + e$ and $q = 10^2a + 10b + c$. Here is the trick:

$$10^4a + 10^3b + 10^2c - (10^2a + 10^b + c) = 9900a + 990b + 99c = 11(900a + 90b + 9c)$$

is a multiple of 11 so that

$$\begin{aligned} q + r &= (10^2a + 10b + c) + 10d + e \\ &= (10^4a + 10^3b + 10^2c) - (10^4a + 10^3b + 10^2c) + (10^2a + 10b + c) + 10d + e \\ &= n - [(10^4a + 10^3b + 10^2c) - (10^2a + 10b + c)], \end{aligned}$$

that is, $q+r$ is n minus a multiple of 11 so that $q+r$ is a multiple of 11 if and only if n is a multiple of 11. So the question becomes: How many multiples of 11 are there in the range 10,000–99,999. There are 89,999 numbers in that range; every eleventh being a multiple of 11 (beginning with 10,010.) Since $89999/11 = 8181.727272\ldots$, the answer is: $\boxed{\text{There are 8,181 such numbers.}}$

Tricky Trimbles. A factory making trimbles has 10 workers, the chief, an experienced older craftsperson and 9 recent young graduates from a Northeastern Trimble University. Each of the nine young workers produces 15 trimbles a day while the chief produces 9 more trimbles per day than the average of all 10 workers.

How many trimbles are produced in a day?

Solution. Here is an easy way to solve this problem. Let us suppose that the chief, being a nice generous person, gives each one of the younger workers one of the 9 trimbles he produced above the day's average, and let's the younger workers claim it as part of their production. This will not change the total production, so the average per worker stays the same also. But now with all young workers producing the same amount and nobody being above the average, all must be producing the same amount of trimbles, namely 16. That means that the day's production is $\boxed{\text{160 trimbles.}}$

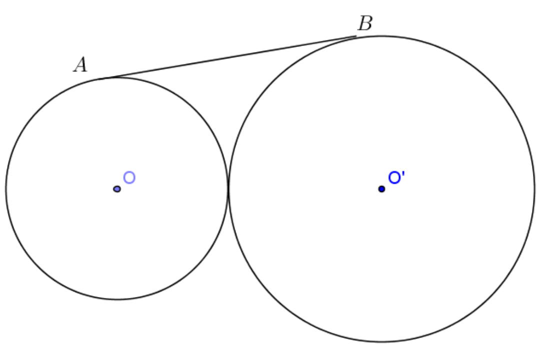
Square Question. How many positive perfect squares less than 1,000,000 (one million) are divisible by 24? Justify your answer. The positive perfect squares are: 1, 4, 9, 16, 25, etc.

Solution. If a square is divisible by a prime, it is divisible by an even power of that prime. So if a square is divisible by $24 = 2^3 \cdot 3$, it must also be divisible by $2^4 \cdot 3^2 = 144 = 12^2$. So all the squares in question will be of the form $144k^2$, where k is a positive integer. We thus want

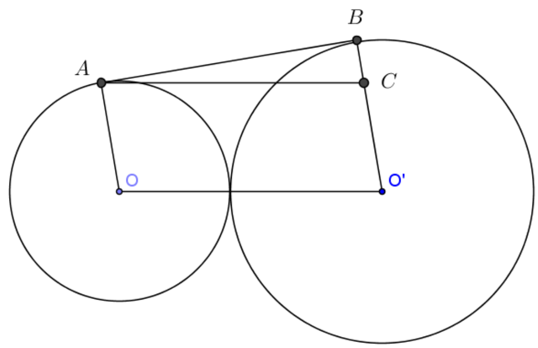
$$k^2 < 1,000,000/144 = 10^6/12^2, \quad thus k < 1000/12 = 83.333\ldots$$

The answer is $\boxed{83}$.

Circulating The circles at centers O, O' are tangent to each other. The segment AB is tangent to both circles. If the radius of the circle of center O is 4, the radius of the other circle is 6, what is the length of the segment AB?



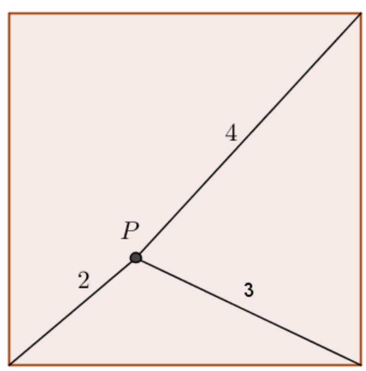
Solution. Draw a segment parallel to OO' from A to the right, intersecting $O'B$ at C.



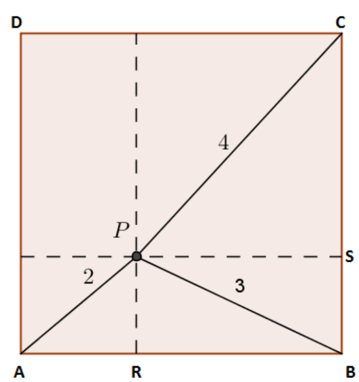
Triangle ABC is a right triangle of hypotenuse AC. Now AC is the sum of the radii, BC the difference of the radii (this needs to be justified, but the justification is easy) so $AC = 10$, $BC = 2$ and

$$AB = \sqrt{10^2 - 2^2} = \sqrt{96} = 4\sqrt{6}.$$

All Praise Pythagoras. A point P inside a square is at distance 2 units from one vertex of the square, at distance 3 units from the next and at distance 4 units from the following one, as shown in the figure below. What is the area of the square? As a hint, there will be a square root in the correct solution.



Solution. Here is a new picture of the square with the vertices given names.



I also drew dashed lines through P intersecting the sides AB and BC perpendicularly at R, S, respectively. Let $\ell = |AB|$ be the length of the side of the square. By Pythagoras,

$$\begin{aligned} |CS|^2 + |BR|^2 &= 16 \\ |BS|^2 + |BR|^2 &= 9 \\ |BR|^2 + |AR|^2 &= 4 \end{aligned}$$

Subtracting the second equation from the first one gets $|CS|^2 - |BS|^2 = 7$. But $|BS| = \ell - |CS|$; from $|CS|^2 - (\ell - |CS|)^2 = 7$ one can solve for $|CS|$ and get $|CS| = \dfrac{7+\ell^2}{2\ell}$. Similarly, subtracting the third equation from the second and using that $|AR| = \ell - |BR|$ one gets $|BR| = \dfrac{5+\ell^2}{2\ell}$. Using these values for $|CS|, |BR|$ in the first equation we get

$$\left(\frac{7+\ell^2}{2\ell}\right)^2 + \left(\frac{5+\ell^2}{2\ell}\right)^2 = 16$$

, which can be rearranged to $\ell^4 - 20\ell^2 + 37 = 0$. Solving this quadratic equation for ℓ^2 we get $\ell^2 = 10 \pm \sqrt{63}$. Choosing for geometric reasons the plus signe we get the answer: The area is $\boxed{10 + \sqrt{63} \text{ units.}}$