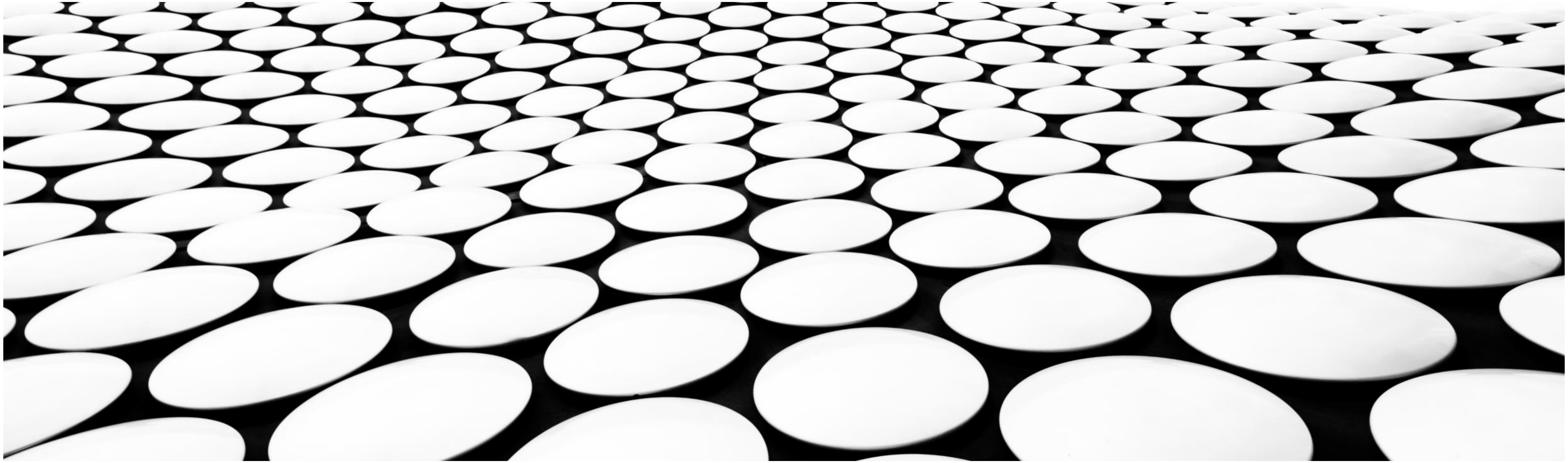
MATH CIRCLE AT FAU

4/13/2024



THE MYSTERY OF PORTIA'S CASKETS

- In Shakespeare's *Merchant of Venice*, the heroine, Portia, has three caskets (boxes) one of gold, one of silver,one of lead. One contained her portrait. Her many suitors were asked to choose one of the caskets; if he chose the one with the portrait, Portia would marry him. On the lid of each casket was an inscription to help the suitor choose wisely. Here we have some variations on this theme. Our Portia wants to marry someone with a logical mind. And here are the questions. (From *What is the Name of This Book*, by R. Smullyan)

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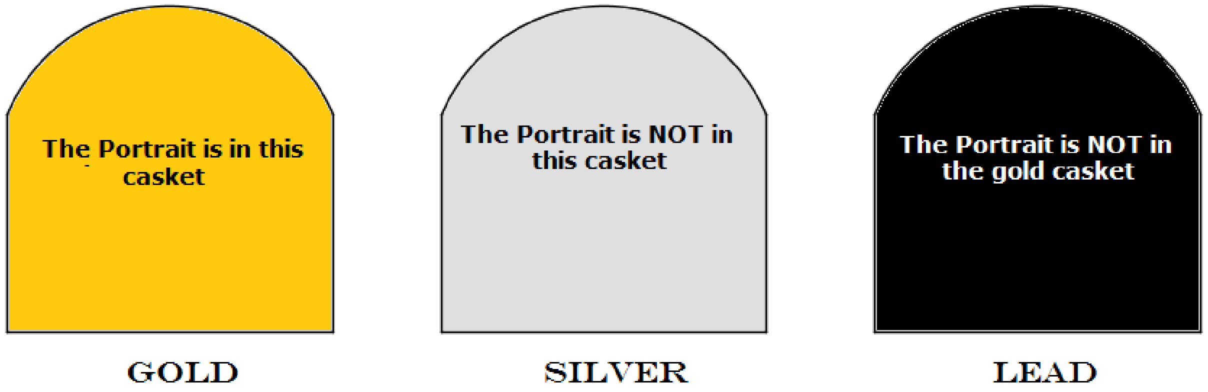
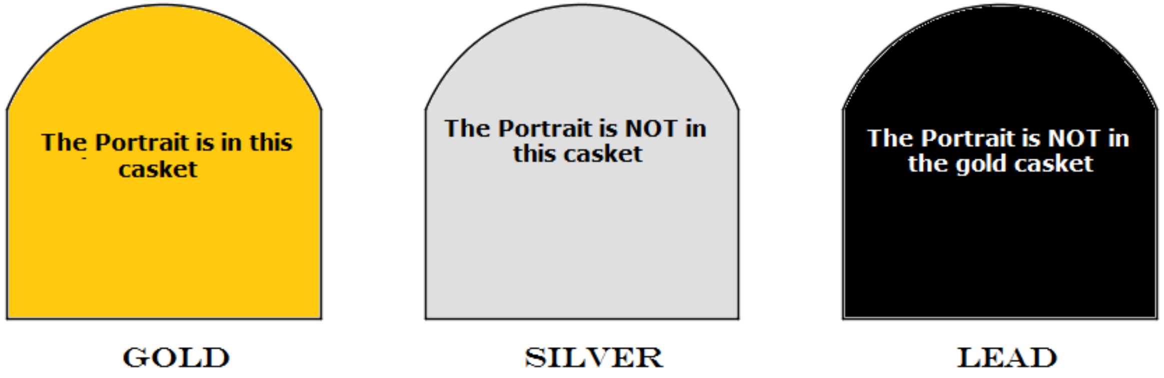


Portia explained to the suitor that of the three statements, at most one was true. Which casket should the suitor choose?

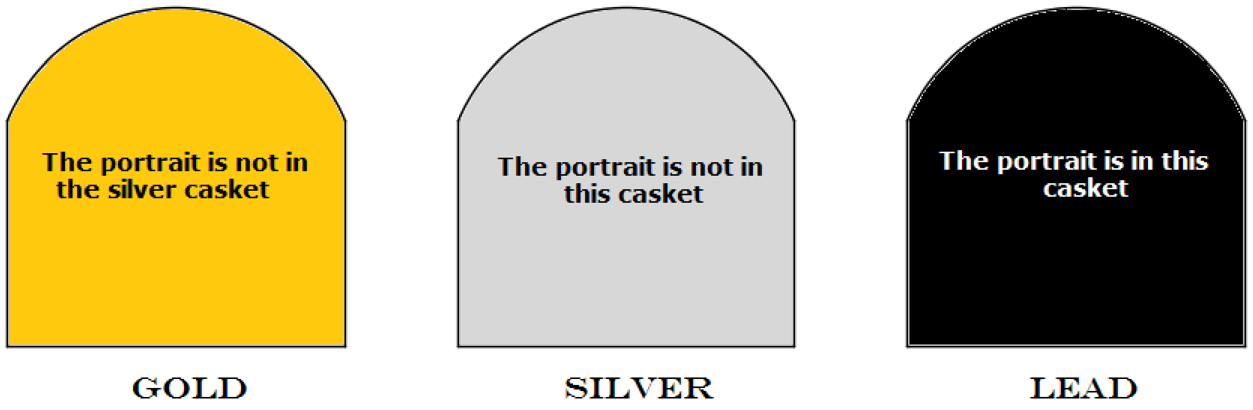
THE MYSTERY OF PORTIA'S CASKETS - SOLUTION

- If Gold is true, then Silver is also true, so Gold is false. But then Lead is true, so Silver must be false. The portrait is in the Silver casket.



THE MYSTERY OF PORTIA’S CASKETS - CONTINUED

- According to Professor Smullyan, Portia married the suitor who chose the right answer, and they were happy for a while. But then she had second thoughts, what if he isn't really that bright? Maybe the puzzle was too easy. So she divorced him and looked for a new husband with a harder puzzle. This time the inscriptions on the caskets were as we can see:



Portia explained that at least one statement was true, and at least one was false. Her ex-husband was one of the new suitors and he guessed correctly. Which casket did he choose?

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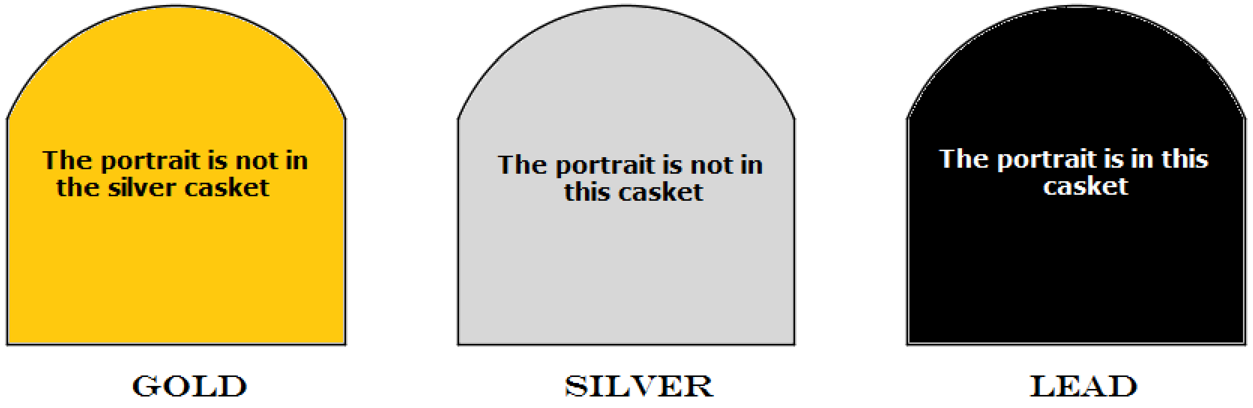
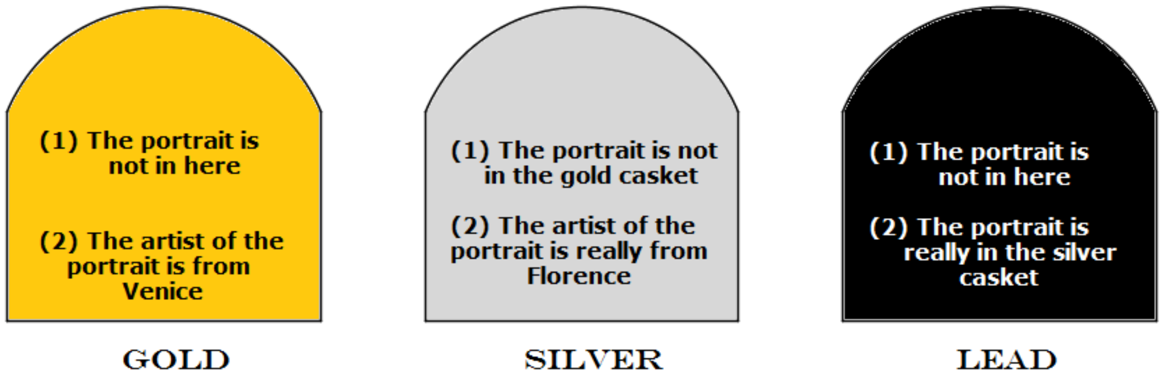


Portia explained that at least one statement was true, and at least one was false. Her ex-husband was one of the new suitors and he guessed correctly. Which casket did he choose?

Suppose the Gold statement false. Then all three statements are false. So G is true. Then S also is true, so L is false. The portrait is in G.

THE MYSTERY OF PORTIA'S CASKETS - CONTINUED

The story goes on that Portia and her logical husband had a daughter, Portia II. When she got to be of marrying age, she decided to do the same as her mother, but better. The suitor had to pass two tests. Here is the first one. The lid of each casket contains two statements, and no lid contains more than one false statement.



Which casket contains the portrait?

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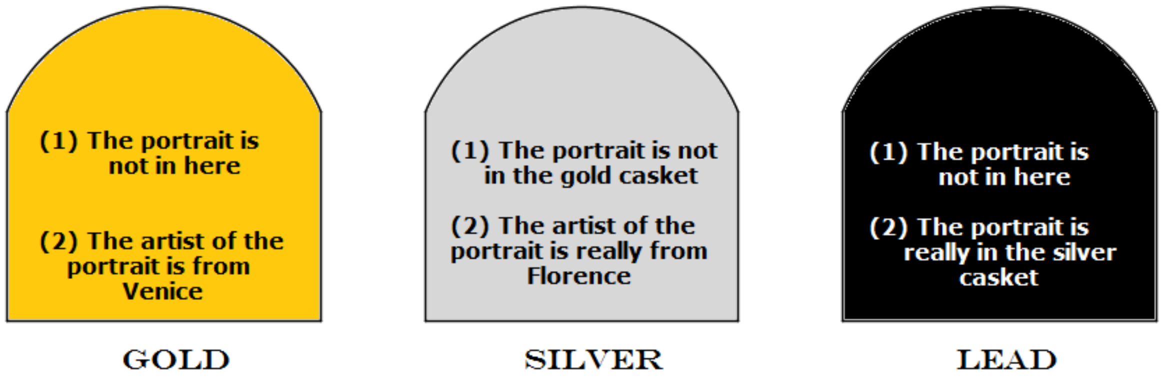
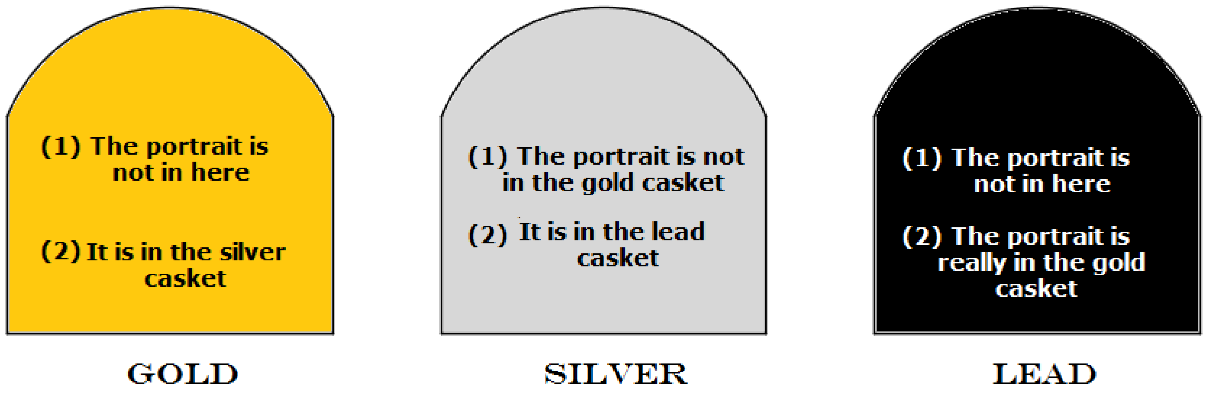


Suppose (2) L is false. Then 1(L) is true and the portrait is in G. But then G(1) and S(1) are false, not possible since one of G(2), S(2) must be false.
So L(2) is true, the portrait is S.

Which casket contains the portrait?

THE MYSTERY OF PORTIA'S CASKETS - CONTINUED

- For the final test, a suitor who passed the previous test, was presented with three more caskets. Portia II explained that on one of the lids both statements were true; on another, both were false; and on the third one statement was true, one was false.



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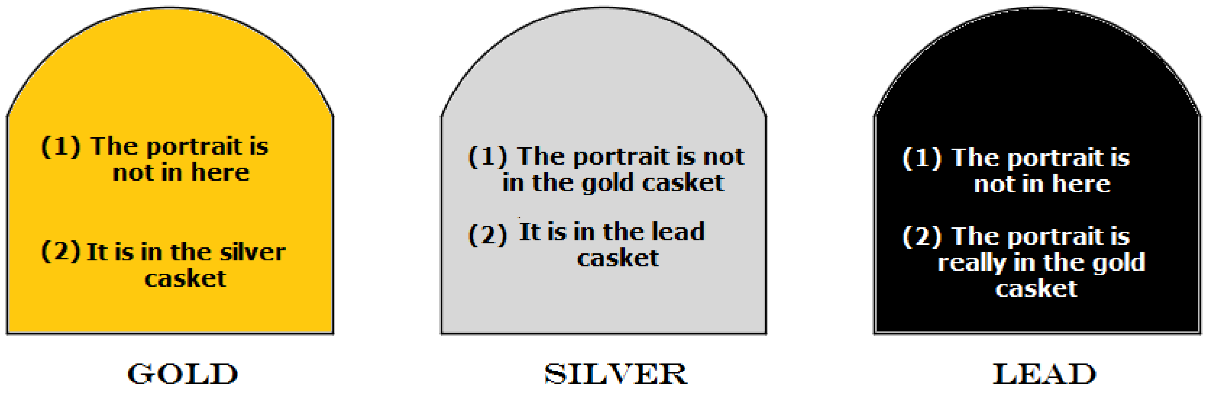

Analysis shows that G has both statements true, S one true, one false, L both false. The portrait is in L.

Which casket contains the portrait

A HERCULEAN TASK

- We have a list of the numbers from 1 to 100

  1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30,

  31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60,

  61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80. 81, 82, 83, 84, 85, 86, 87, 88, 89, 90,

  91, 92, 93, 94, 95, 96, 97, 98, 99, 100

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61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80. 81, 82, 83, 84, 85, 86, 87, 88, 89, 90,

91, 92, 93, 94, 95, 96, 97, 98, 99, 100

Suppose one does the following:
Take any two numbers x, y from the list, cross them out and add to the list the number $x + y + xy$, Repeat. Repeat.

Your task is to answer the following questions, with reasons:
1. How many steps will it take before one is reduced to a single number?
2. The final number, is it always the same? or does it depend on the order of crossing out and replacing?
3. Supposing it is always the same, what is it? If not the same, what numbers can be final numbers?

A HERCULEAN TASK - SOLUTION

- In 99 steps, no matter in what order one proceeds, one ends with a single number, namely

$$101! - 1 = 9425947759838359420851623124482936749562312794702543$$
$$7683278893534169775993162214765030878615918083469116$$

$$101! - 1 = 9425947759838359420851623124482936749562312794702543 \\ 768327889353416977599316221476503087861591808346911 \\ 623490003549599583369706302603264000000000000000000000000$$

Eight friends ate at a restaurant and agreed to share the bill equally. Because Judi forgot her money, each of her seven friends paid an extra $2.50 to cover her portion of the bill. What was the total bill?

EATING OUT

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Solution: $ 140

EATING OUT

A YEARLY QUESTION

- 2024 can be obtained as the sum of 23 consecutive integers. What is the first and last of these integers?

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- SOLUTION: The first number is 77, the last 99.

$$77 + 78 + \cdots + 99 = 2024.$$

JUMPING JINKS

Floppy, the magic rabbit, can double the distance it can jump with each successive jump. Its first jump is 1 foot, its second 2 feet, its third 4 feet, and so on. On which jump will it first be able to jump more than a mile? A mile equals 5280 feet.

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Solution: At the 14th jump

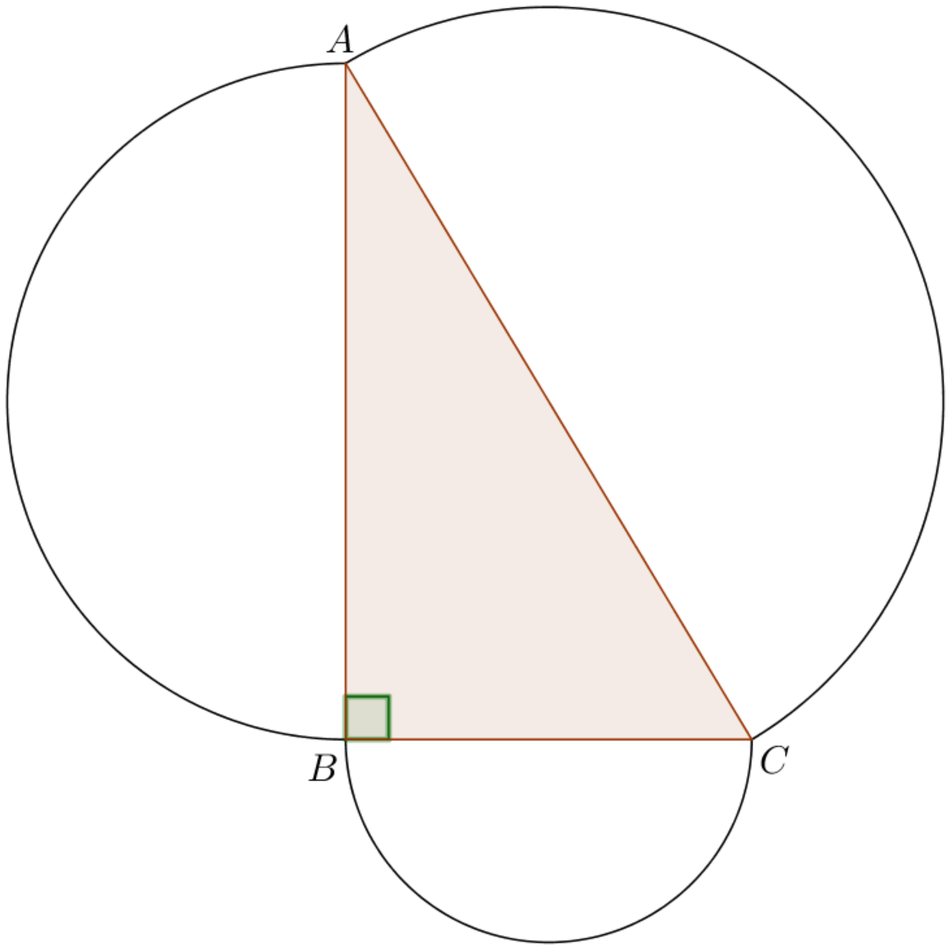
TRIANGLES ARE FREQUENTLY RIGHT

The angle at B of triangle ABC pictured below is a right angle. The sides of triangle ABC are the diameters of half circles, as pictured to the right. The arc of the semicircle on AB has a length of 7π.

The area of the semicircle on AC equals 29π.

What is the radius of the semicircle on BC?

Note: The picture is not in scale.



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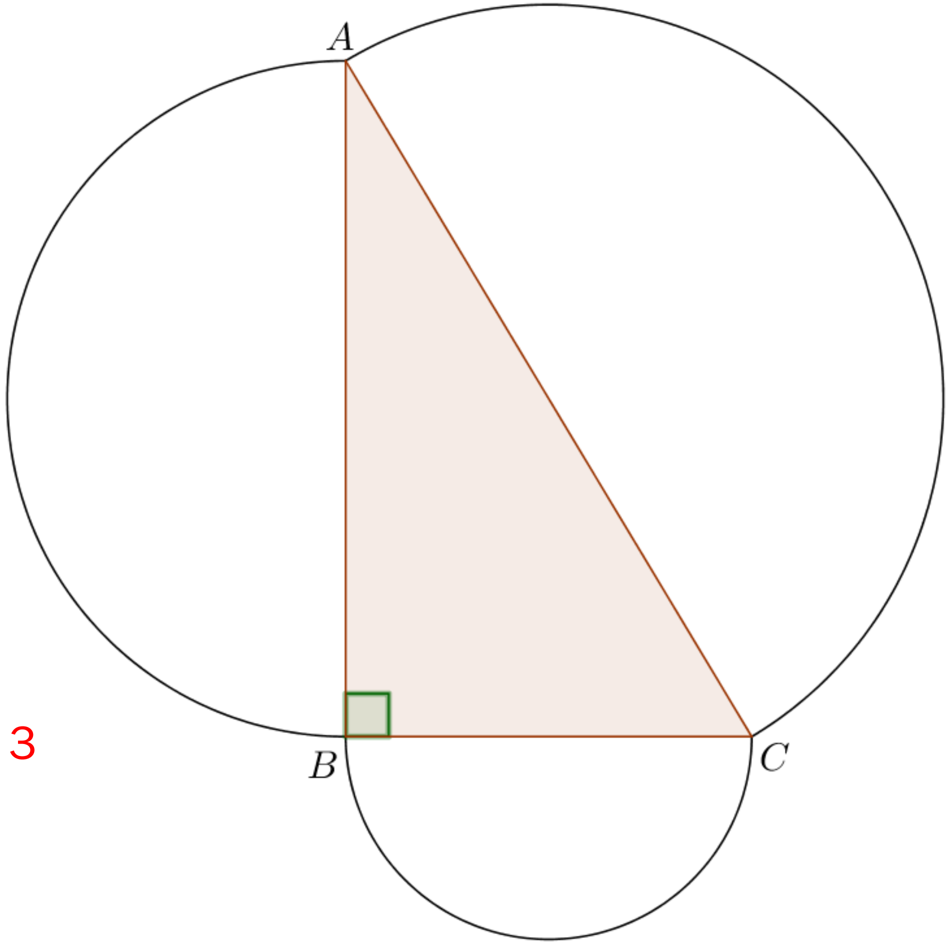
Solution: 3



STRANGE WORLDS

- In the country of Nosuchplacia, by decree of the emperor Ineptus III, all purchases have to be paid in cash.There is no credit, no checks; online does not exist. The currency unit of Nosuchplacia is the dollarium. However, the Nosuchplacia Treasury only prints bills in denominations of 9 dollariums and 12 dollariums. Describe all prices that items can have in a store if

(a) The store can give change.

(b) The store does not give change.

(c) After many pleas, Ineptus III kept the 12 dollarium bills, but replaced all 9 dolarium bills with 7 dolarium bills. Answer the same questions as above,

STRANGE WORLDS - SOLUTION

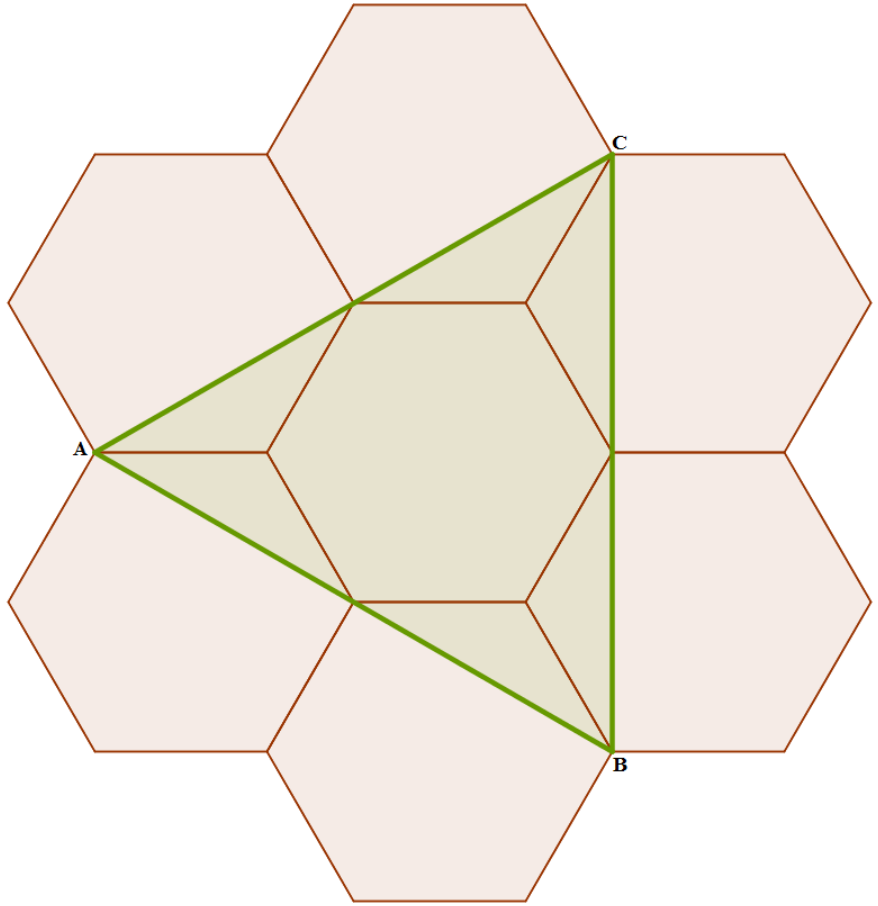
(a) Because the common greatest divisor of 9 and 12 is 3, only process that are multiples of 3 are possible. If the store gives change, all such process can be obtained.

(b) If no change is given, the only possible prices are 9, 12, and all multiples of 3 greater than or equal 21.

(c) Since 7 and 12 have greatest common divisor 1, all prices are possible, with change. Without change the situation is quite complicated but from a certain point onward all prices aer possible. I leave it as an exercise to find that point.

HEXAGONS ON HEXAGONS

Six regular hexagons surround a regular hexagon of side length 1, as shown in the picture below. What is the area of triangle ABC?



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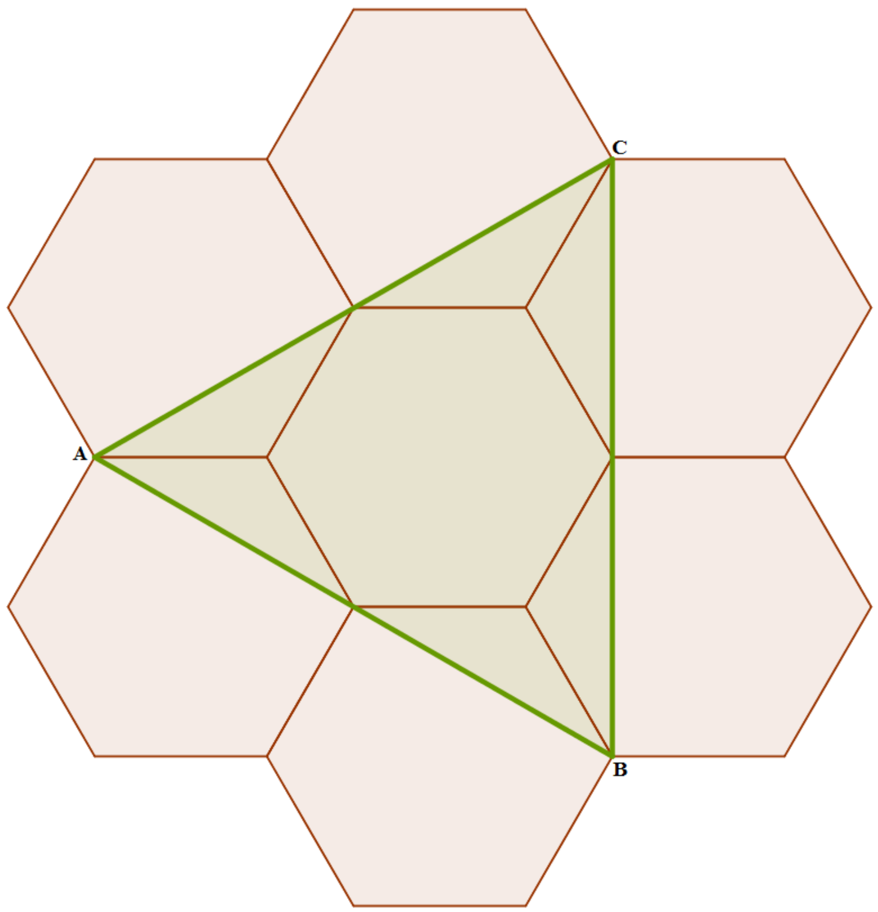
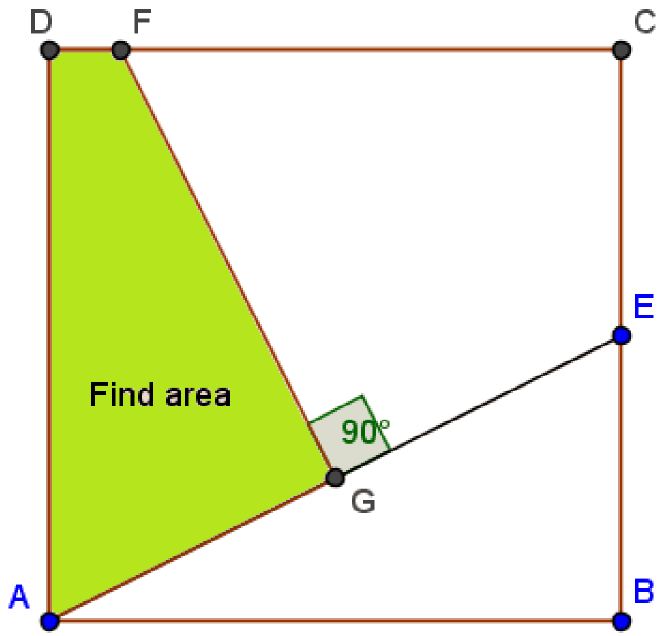
Solution: $3\sqrt{3}$



MORE AREAS

In the square $ABCD$, E is the midpoint of the side BC and G is the midpoint of the segment AE. The segment GF is perpendicular to AE. If the area of the square $ABCD$ is 16, find the area of the quadrilateral $AGFD$.



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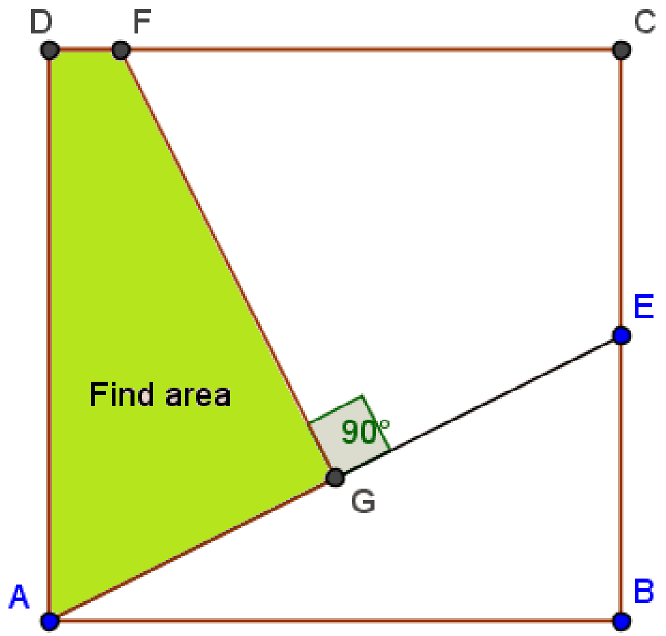

Left as homework.