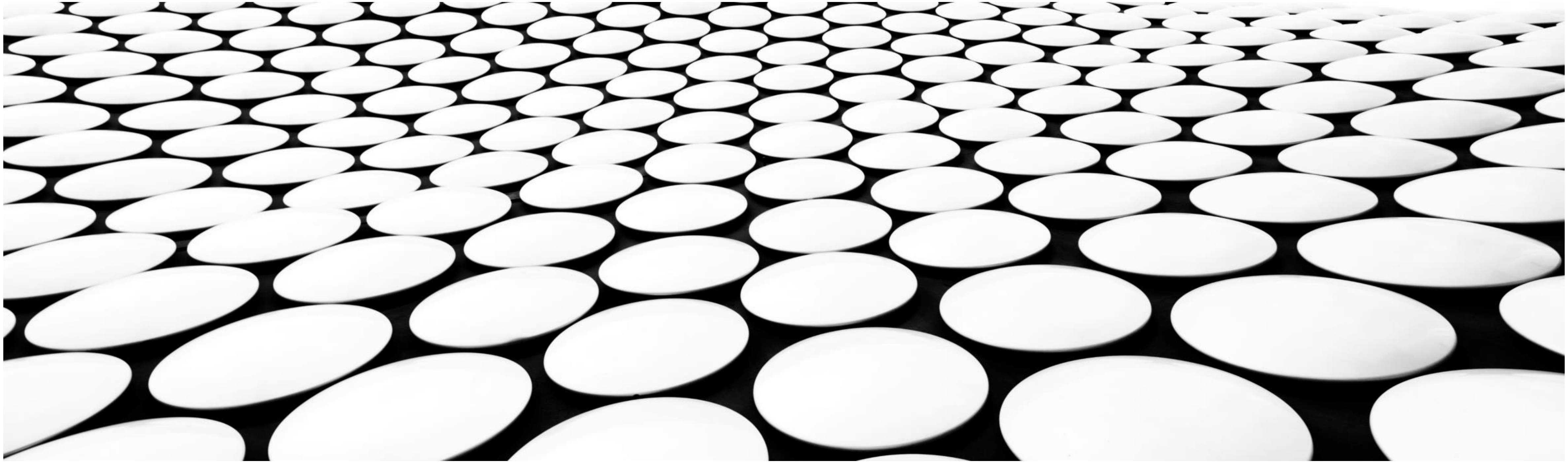
MATH CIRCLE AT FAU

3/16/2024



THE ISLAND OF KNIGHTS AND KNAVES

Somewhere in some far away sea, or perhaps in the Bermuda Triangle, lies the island of knights and knaves, discovered (some say invented) by the great logician Raymond Smullyan. All inhabitants of all genders and ages are either knights or knaves. There is no telling which is which from just looking at them, But knights always tell the truth; they are unable to lie even to save their lives. Knaves always lie; always.

KNIGHTS ARE ALWAYS TRUTHFUL, KNAVES ALWAYS LIE.

- We are going to look at some statements and your task will be to see if you can classify the speakers as knights or knaves. The statements are all from Raymond Smullyan's *Gödelian Puzzle Book.*

THE ISLAND OF KNIGHTS AND KNAVES

On a visit you meet three inhabitants: A, B, and C.
You are told that at least one is a knight, one is a knave, and one holds a prize you can have if you figure out who has it.

Here is what they say:

A: B doesn't have the prize.
B: I don't have the prize.
C: I have the prize.

What are A, B, C, and who has the prize?

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Here is what they say:

A: B doesn't have the prize.
B: I don't have the prize.
C: I have the prize.

What are A, B, C, and who has the prize?

Answer: A, B are knights, C is a knave and A has the prize.

NEXT ADVENTURE

- On another visit you meet three other inhabitants A, B, C. You were reliably informed that one of them is a magician. Here is what they say.

A: B is not both a knave and a magician.

B: Either A is a knave, or I am not a magician.

C. The magician is a knave.

What are A, B, C, and who is the magician?

NEXT ADVENTURE - ANALYSIS

- Let us assume first A is a knight. Then what A says is true; another way of saying the same thin is: Bis a knight or B is not a magician. That makes true what B says; ***an OR statement is true if one of its clauses is true.*** So B Is a knight. We get to C.
- If C is also a knight, then all are knights (possible), but then what C says is false. So C must be a knave and the magician is a knight. Coming back to B we decided on B’s knighthood based on B saying, “I am not a magician” The only remaining possibility is that A is the magician. So :
- A, B are knights, C is a knave, A is the magician works perfectly.
- But is there no other choice? Suppose A is a knave, so A lies. Then B is both a knave and a magician. But this conflicts with the first clause in B’s OR statement, which would be tru if A is a knave. So the possibility of A a knave is out.
- A, B are knights, C is a knave, A is the magician

THE PLOT THICKENS

- On another visit you meet three married couples, the Arks, the Bogs and the Cogs. One of the three couples is the king and queen of the island. I was informed that in each couple at least on member was a knight. Here is what they said:

- Mr. Ark I am not the king.
- Ms. Ark The king was born in Italy.

- Mr. Bog Mr. Ark is not the king.
- Ms. Bog The king was really born in Spain.

- Mr. Cog I am not the king.
- Ms. Cog Mr. Bog is the king.
- Who is the king?

THE PLOT THICKENS - ANALYSIS

- On another visit you meet three married couples, the Arks, the Bogs and the Cogs. One of the three couples is the king and queen of the island. I was informed that in each couple at least on member was a knight. Here is what they said:

- Mr. Ark I am not the king.
- Ms. Ark The king was born in Italy.

- Mr. Bog Mr. Ark is not the king.
- Ms. Bog The king was really born in Spain.

- Mr. Cog I am not the king.
- Ms. Cog Mr. Bog is the king.

- Who is the king?

Mr. Cog must be telling the truth, since otherwise Ms. Cog would also lie. So Mr. Cog is a knight and not the king. Ims. Cog is telling the truth we are done, Mr. Bog is the king. But is this contradiction free? One thing to see, regardless of Ms. Cog's type, is that once we know that either Mr. Bog or Mr. Ark is the kning, then Mr. Bog has to be a knight. In fact, if a knave, then Mr. Ark is also a knave. Then both Ms. Ark and Ms. Bog have to be knights; impossible since they make contradictory statements. The conclusion is that Mr. Bog is the king.

We can also establish that Mr. Ark and Ms. Cog are knights. The types of Ms. Ark and Ms. Bog are inconclusive, but not both are knights.

THE INTERROGATION

- Three island couples, the Dags, the Eggs and the Fens, were being interrogated because it was known that one of the members was was a spy. It is known that in one of the couple both man and woman are knights, in another couple both members are knaves; in the third couple one is a knight, the other a knave. The couples stated.

- Mr Dag: I am not the spy.   Ms. Dag: Mr. Egg is the spy.
- Mr. Egg: Mr. Dag is truthful   Ms. Egg: Mr. Fen is the spy.
- Mr Fen: I am not the spy.   Ms. Fen: Mr Dag is the spy.

- Who is the spy?

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- Mr Dag: I am not the spy. Ms. Dag: Mr. Egg is the spy.
- Mr. Egg: Mr. Dag is truthful Ms. Egg: Mr. Fen is the spy.
- Mr Fen: I am not the spy. Ms. Fen: Mr Dag is the spy.

- Who is the spy?

The only consistent solution is that the Eggs are both knights, the Fens are both knaves and Mr. Dag is a knight, Ms. Dag a knave. Nr. Fen is the spy.

WITCH DOCTORS

A visitor to the island, who ventured into the forbidden zone, was captured ny a ferocious band of brigands. The visitor was shown three islanders, A, B, and C, and told that one was a witch doctor. The visitor was told to point at one of the trio. If the visitor pointed to the witch doctor, the visitor was to be executed. Otherwise, the visitor could leave in peace. Here is what A, B, C said:

A: I am the witch doctor.
B: I am not the witch doctor.
C: At most one of us is a knight.

To which of A, B, C should the visitor point?

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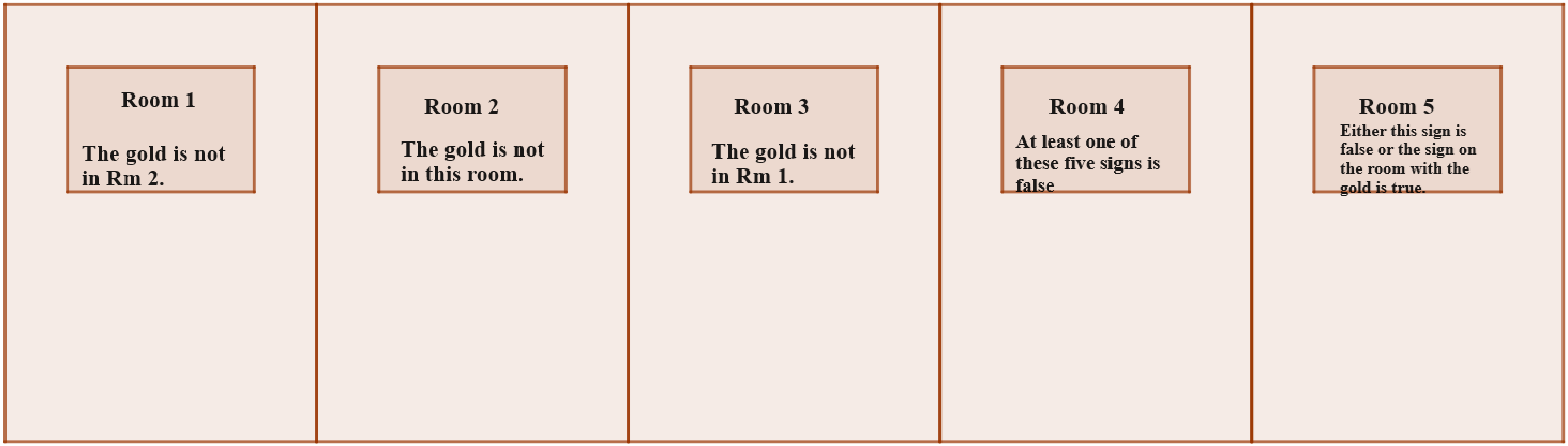
A: I am the witch doctor.
B: I am not the witch doctor.
C: At most one of us is a knight.

To which of A, B, C should the visitor point?

Of A, B, C, C is the only one that cannot be the witch doctor. The visitor should point to C.

THE POT OF GOLD



On the island, a hall has five adjacent doors, one leads to a room with a pot of gold. Each door has a sign. Which door leads to the gold, and which signs are true.

POT OF GOLD - ANALYSIS

The Statements are:
Rm 1: The gold is not in Rm 2.
Rm 2: The gold is not in this room.
Rm 3: The gold is not in Rm 1
Rm 4: At least one of the signs is false.
Rm 5; Either this sign is false or the sign on the room with the gold is true.

An or statement is false ONKLY if both parts are false. For the statement on Rm 5 to be false false, it must be true. This being contradictory, the sign on 5 is true. Since it is false it is false, the pert that says that the sign on the door of the gold room is true, must be true.

If the sign on 4 is false, then all signs are true, including the one on 4. So the sign on 4 must be true.

Because 5 is true, the sign on the gold room is true. But then 2 is not the gold room and the sign on 2 is true. But now 1 is also true. Since there has to be a false sign, it must be 3 and the gold is in Rm 1.

And now for something completely different

BUILDING TRIANGLES

- If one side of a triangle is 3.8 inches long, another side is 0.6 inches long, how long is the third side if its length is a whole number of inches?

BUILDING TRIANGLES

- If one side of a triangle is 3.8 inches long, another side is 0.6 inches long, how long is the third side if its length is a whole number of inches?

- Answer: 4 inches.

SIDES VS. DIAGONALS

- In a square, the sum of the lengths of all four sides is larger than the sum of the lengths of the diagonals. Is that true for all quadrilaterals? Why?

SIDES VS. DIAGONALS - ANSWER

- In a square, the sum of the lengths of all four sides is larger than the sum of the lengths of the diagonals. Is that true for all quadrilaterals? Why?

- If the quadrilateral is ABCD and the diagonals are AC and BD, then looking at the triangle ABC we conclude

$$AC < AB + BC$$

- Looking at triangle ACD we see

$$AC < AD+DC$$

- Adding: $2AC < AB+BC+CD+DA$. Similarly, $2BD < AB+BC+CD+DA$, Adding again and dividing by 2

$$AC+BD < AB+BC+CD+DA.$$

- The answer is Yes, always.

NUMBER SUMS

- Is it possible to write more than 50 two-digit numbers on a blackboard without having two numbers on the board adding up to 100?

NUMBER SUMS - SOLUTION

- Is it possible to write more than 50 two-digit numbers on a blackboard without having two numbers on the board adding up to 100?
- No.

- Let us write down all the pairs that add up to 50. They are

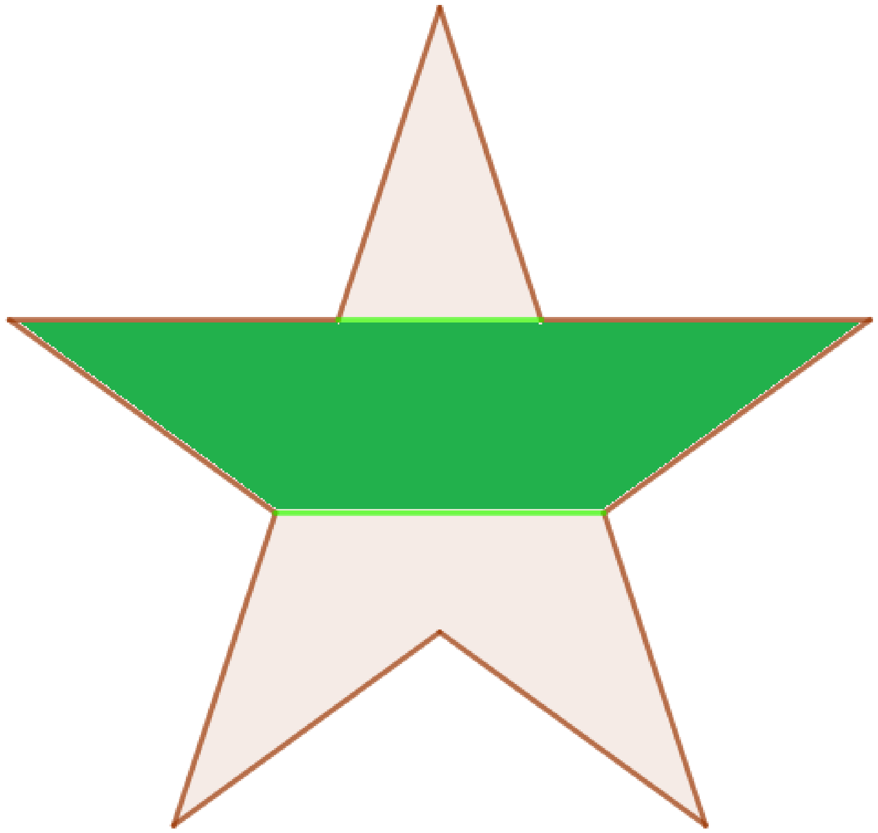
(10,90), (11, 89), (12, 88), … , (48,52), (49,51),

40 pairs in all. We also have the 10 numbers that have no pair: 50, 91, 92, …, 99.

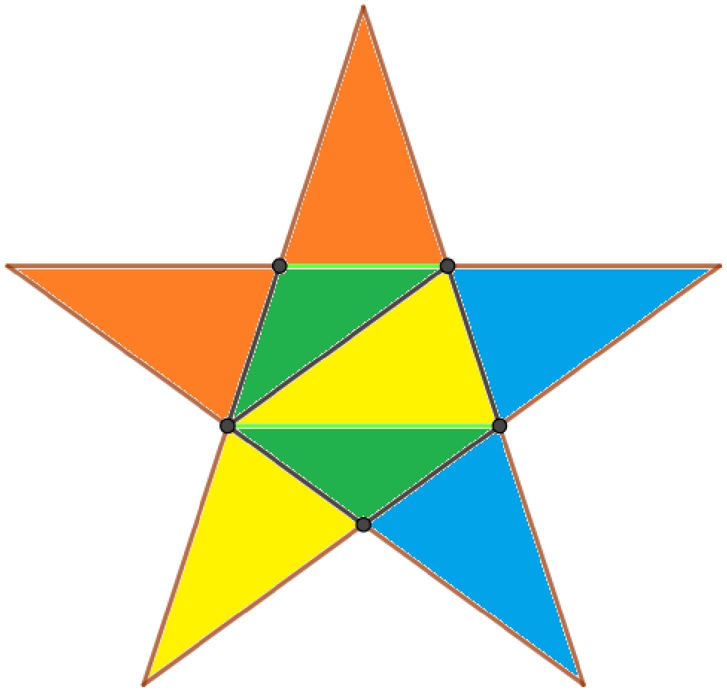
In writing down 50 numbers trying to avoid a sum of 100, we have to take at most one number from each pair (40 numbers at most) and numbers from the isolated 10. That gives us at most 50 numbers. To get to 51 we have to include both numbers from one of the pairs.

PENTAGRAMS

Show that the area of the green pentagon in the picture is exactly half the area of the regular pentagonal star



PENTAGRAM



IS THERE SUCH A NUMBER?

- Is there a number of the form

$$11\ldots10\ldots0$$

That is divisible by 2024?

- Why?

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$$11 \ldots 10 \ldots 0$$

That is divisible by 2024?

- Why?

- The answer is Yes. You need to figure out why.