Math Circle at FAU
February 10, 2024
The game of Set and Combinatorics

1 Attributes

Each card in a deck of Set has 4 so called *attributes*

1. **Shape:** The shapes are ovals, diamonds, or squiggles.
2. **Number:** In each card there are either 3 equal shapes, 2 shapes, or just one shape.
3. **Color:** There is only one color per card; green, purple, or red.
4. **Shading** Empty (only the outline of the shapes are colored), stripes, or solid.

2 Sets

A *set* consists of three cards that are either all the same or all different in each of the four attributes.

All cards are different. Given this information, here is your first question.

Question 1 *How many cards are there in a deck of Set?*

3 Coordinates

We will use initial G, P, R for the colors (green, purple, red, respectively), the shapes will be identified by O, D, Sq (diamonds, ovals, squigglies); the shadings by E, St, So (empty, striped, solid). Then every card can be identified by a number and a string of three of these letters. The order is sort of arbitrary, but will be number, color, shading, shape. For example (3,G,E,Sq indicates the card having three green squiglies, with empty interior, (2,R,So,D) indicates the card with two solidly red diamonds.

4 The fundamental Theorem of Set

Given any two cards, there is a unique third card completing the set.

As an exercise, complete the set that has the following two cards: (2,G,St,O), (2,G,So,O).

5 More Questions

Question 2 *How many different sets are there?*

Question 3 *How many sets contain card (1,G,E,O)?*

Question 4 *How many sets have all four attributes different?*

Question 5 *How many sets have exactly three attributes different? So one the same, the other three different.*

Question 6 *Two the same, two different?*

Question 7 *How many sets with three attributes the same, one different?*

Question 8 *Can you write down a good reason why the following statements are true? Are they?*

- *The number of ways we can choose 3 elements from a set of 4 equals the number of ways we can choose 3 elements from a set of 3 plus the number of ways we can choose 2 elements from a set of 3.*

  This could be done by just checking. So if the set is $\{a, b, c, d\}$, the way we can choose three elements are

  $$\{a, b, c\},\ \{a, b, d\}, \{b, c, d\},$$

  three ways. It is then easy to check that there really is only one way to choose 3 elements from a set of 3; namely choose all. There are 2 ways of choosing 2 elements from a set of 3. And $1 + 2 = 3$. But is there a better way?

- *The number of ways we can choose 5 elements from a set of 7 equals the number of ways we can choose 5 elements from a set of 6 plus the number of ways we can choose 4 elements from a set of 6.*

  Not so easy to do by checking. Still possible.

- *The number of ways we can choose 57 elements from a set of 100 equals the number of ways we can choose 57 elements from a set of 99 plus the number of ways we can choose 56 elements from a set of 99.*

  Good luck checking this directly!

A standard symbol used in mathematics for the number of ways we can choose k elements from a set of n (also called "n choose k"), is C^n_k or, in more sophisticated circles $\left(\begin{array}{c} n \\ k \end{array}\right)$. Since we are sophisticated, I'll use $\left(\begin{array}{c} n \\ k \end{array}\right)$. One defines, perhaps arbitrarily, $\left(\begin{array}{c} n \\ 0 \end{array}\right) = 1$; there is only one way you can choose nothing from a set. Meditate on this. It makes the formulas work.

Question 9 *Fill in the blanks. Here n, k are positive integers.*

$$\begin{aligned}
\binom{n}{1} &= __ \\
\binom{n}{n} &= __ \\
\binom{n-1}{k} + \binom{n-1}{k-1} &= __ \\
\binom{n}{k} &= __ \quad \textit{if} \quad k > n
\end{aligned}$$