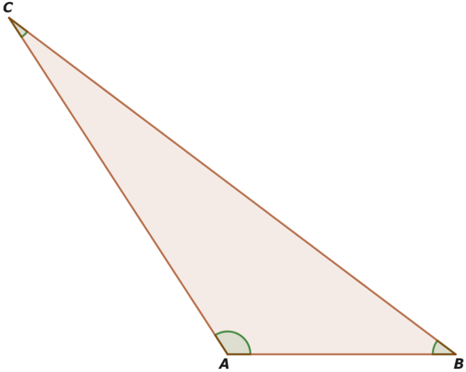
Geometry Basics and More

Math Circle at FAU

October 14, 2023

Outline

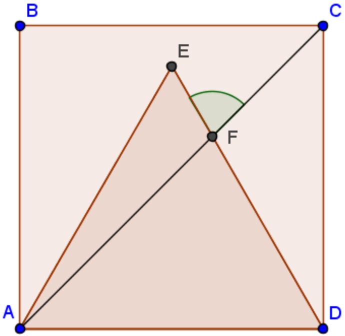
Basic Question



- In the triangle pictured above,
  - The measure of the angle at B is 37°.
  - The measure of the angle at C is 20°
- What is the measure of the angle at A?

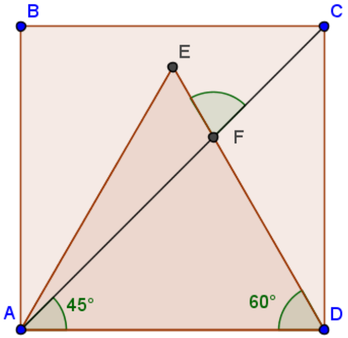
Angling for angles

- The figure below shows equilateral triangle AED inside square $ABCD$. The segment AC is a diagonal of the square. What is the measure of $\angle EFC$?



Solution

Because the triangle AED is equilateral, all of its angles measure 60°. Since the segment CA bisects a right angle, $\angle DAC$ measures 45°.

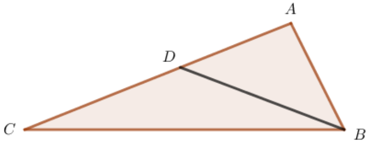


ANSWER: 75°.

The two base angles of triangle AFD add up to $45^\circ + 60^\circ = 105^\circ$, that leaves $\angle AFD$ no choice but to measure $180^\circ - 105^\circ = 75^\circ$. A very basic property of angles is that if two lines cross each other, then opposite angles are equal. Thus $\angle EFC$, and $\angle AFD$, being opposite angles created by the crossing of lines AC and DE, must be equal. Since $\angle AFD$ measures 75°, so does $\angle EFC$.

Angling for Angles

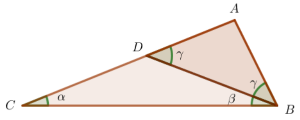
In $\triangle ABC$ the point D is on AC is such that $|AB| = |AD|$. If $\angle ABC - \angle ACB = 30^\circ$, find $\angle CBD$. Justify your answer.



Solution

It may be convenient to give some names to the angles. It is traditional to use Greek letters for this purpose. Notice that because $\triangle ABD$ is isosceles, $\angle ABD = \angle ADB$. In the picture below I renamed the angles by

$$\alpha = \angle ACB, \quad \beta = \angle CBD, \quad \gamma = \angle ABD = \angle ADB.$$

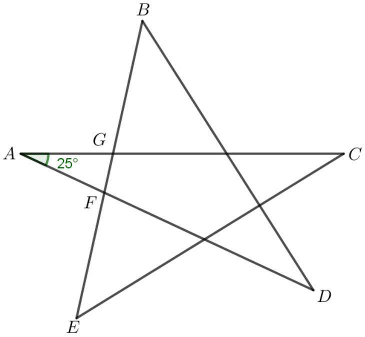


We can rephrase the problem by: Find β given that $(\beta + \gamma) - \alpha = 30$. Using that the sum of two angles of a triangle equal the angle supplementary to the third angle, we see that $\alpha + \beta = \gamma$ or $\beta = \gamma - \alpha$. From the given equation, $\beta = 30 - (\gamma - \alpha)$. Thus

$$2\beta = (\gamma - \alpha) + 30 - (\gamma - \alpha) = 30, \text{ so } \boxed{\beta = 15^\circ.}$$

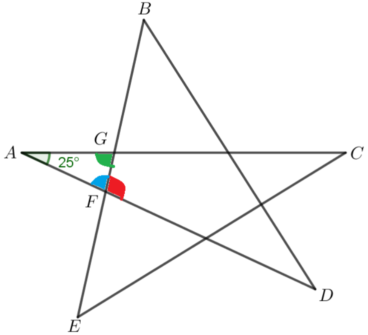
The Angle is in the Star

In the star shaped figure, the angle at A measures 25° and $\angle AFG = \angle AGF$. Find $\angle B + \angle D$. (AMC 8)



Solution

- $\angle B + \angle D = 180^\circ$ - red angle;
- blue angle $= 180^\circ$ - red angle.
- Thus $\angle B + \angle D =$ blue angle.
- Because $\triangle AFG$ is isosceles, blue angle $=$ green angle.
- $180^\circ = 25^\circ +$ blue angle $+$ green angle $= 25^\circ + 2\times$ blue angle.
- $\angle B + \angle D =$ blue angle $= \frac{1}{2}(180 - 25) = \boxed{77.5^\circ}$.



A Counting Intermezzo

- The integers 234, 417, 645 share a curious property: All three digits are different and one of the three digits is the average of the other two. How many three-digit numbers have this property? That is, how many three digit numbers are composed of three **distinct** digits such that one digit is the average of the other two?

Solution

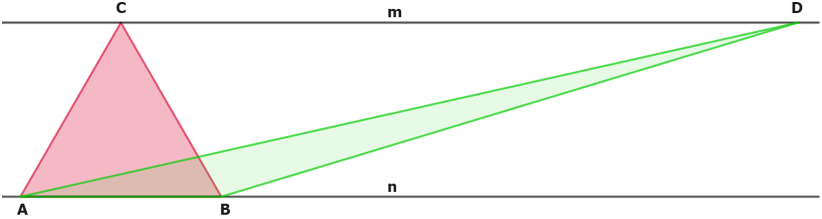
Let us first get all sets of three digits $\{a, b, c\}$ such that $c = (a+b)/2, a < b$. For c to be a digit, either a, b are both even (10 choices) or both odd (also 10 choices). We have 20 choices in all. Each one of these choices can be arranged in 6 different ways. For example, from $(3, 5, 4)$ we get the integers

$$345, 354, 453, 435, 534, 543.$$

This gives a total of $6 \times 20 = 120$ integers. But some of these will have a leading 0! If $a = 0$, then b is even, $c = b/2$, so we have to disregard the integers $021, 012, 042, 024, 036, 063, 084, 048$, eight integers in all. The final answer is that there are 112 such integers.

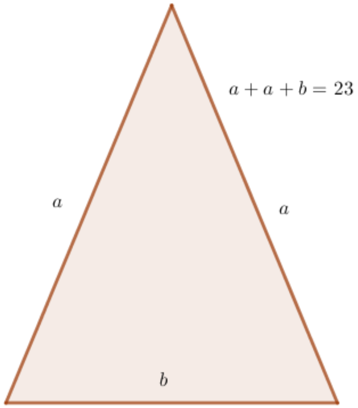
Who Wins?

- Lines m, n are parallel.
- Which triangle has the larger area? $\triangle ABC$ or $\triangle ABD$?



Isosceles Inquiries

- How many different isosceles triangles have integer side lengths and perimeter 23? (AMC 8)



Solution

If the sides of the triangle are a, a, b then a, b must satisfy $2a > b$ and $2a + b = 23$. All choices of a, b satisfying these conditions work. We have

$$23 = 2a + b < 4a, \quad \text{so} \quad a > 23/4 = 5.75.$$

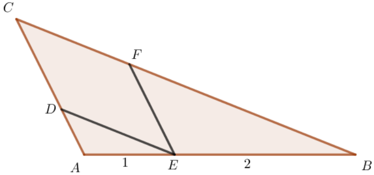
Being an integer $a \geq 6$. next, since $b \geq 1$,

$$23 = 2a + b \geq 2a + 1, \quad \text{so} \quad 2a \leq 22,$$

hence $a \leq 11$. So a is one of $6, 7, 8, 9, 10, 11$ giving a total of 6 such triangles.
Now $2a > b, 2a + b = 23$ implies $2a \geq \lceil 23/2 \rceil$, so $a \geq 6$. We also must have $a \leq 11$. But there are no other restrictions on a so that we have a total of 6 such triangles.

The Parallelogram Intruder

- Another AMC 8 problem.
- In triangle ABC point E is on AB with $|AE| = 1, |EB| = 2$. Point D is on AC so that $DE \| BC$ and point F is on BC so that $EF \| AC$. What is the ratio of the area of $CDEF$ to the area of $\triangle ABC$?



Solution

$\triangle EBF$ is similar to $\triangle ABC$. Because $|EB|/|AB| - 2/3$, the constant of proportionality is $2/3$ and $[EBF] = (2/3)^2[ABC]$. Similarly, $\triangle AED \sim \triangle ABC$; since $|AE|/|AB| = 1/3$, we get $[AED] = (1/3)^2[ABC]$. Then

$$[ABF]+[AED] = (\frac{4}{9}+\frac{1}{9})[ABC]; \quad \text{that is,} \quad [ABF]+[AED] = \frac{5}{9}[ABC].$$

Now

$$[CDEF] = [ABC] - ([ABF] + [AED]) = \frac{4}{9}[ABC].$$

The answer is $\boxed{4/9.}$