Quantitative Analysis of Attractors in Finite-Dimensional Approximations of State-Dependent Delay Differential Equations

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Abstract

In this work we study, we study a certain class of delay differential equations (DDEs) which we refer to as state dependant delay maps. These are DDEs where the right hand side depends only on the delayed variables (that is, where the undelayed variables absent from the equation). A delay map with constant delays can be written explicitly as a discrete time dynamical system on an appropriate function space, and a delay map with small state dependent terms can be viewed as a "non-autonomous" perturbation. We develop a fixed point formulation for the Cauchy problem and under appropriate assumptions obtain the existence of forward iterates of the map. The proof is constructive and leads to numerical procedures which we implement for some illustrative examples.

1 Introduction

[2, 3, 1]

Suppose that $f\colon \mathbb{R}^2 \to \mathbb{R}$ and $\tau > 0$. The equation

$$y'(t) = f(y(t), y(t-\tau)) \tag{1}$$

is a delay differential equation with constant delay. Since the value of the derivative depends on both the current value of the function and it's value at τ time units "in the past", it is necessary to specify a history function for the problem. Suppose then that $y_0\colon [-\tau, 0] \to \mathbb{R}$ is a given, continuous function. We

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are interested in the existence of a function $y\colon [-\tau, T] \to \mathbb{R}$ so that $y(t) = y_0(t)$ for $t \in [-\tau, 0)$ having that $y(t)$ solves Equation (1) on $[0, T]$.

A particularly simple case is when f depends only on the past. In this case the equation reduces to

$$y'(t) = f(y(t - \tau)), \tag{2}$$

with y_0 given. Integrating both sides of Equation (2), and substituting the history function into the integrand leads to

$$\begin{aligned} y(t) &= y(0) + \int_0^t f(y(s - \tau))\, ds \\ &= y_0(0) + \int_{-\tau}^{t-\tau} f(y_0(s))\, ds \qquad\qquad t \in [0, \tau]. \end{aligned}$$

In this case $y(t)$ exists and is unique on $[0, \tau]$, and is differentiable on $(0, \tau)$ as long as $y_0(t)$ is continuous.

Indeed, by shifting result back to the interval $[-\tau, 0]$ we define dynamical system on $C([-\tau, 0], \mathbb{R})$ by

$$\Phi[y](t) = y(0) + \int_{-\tau}^{t} f(y(s))\, ds.$$

We refer to Φ as a delay map. To justify the terminology and define the orbit of $y \in C([-\tau, 0], \mathbb{R})$ by

$$y_{n+1}(t) = \Phi[y_n](t),$$

for $n = 0, 1, 2, \ldots$, where $y_0(t) = y(t)$. If $\{y_n\}_{n=0}^{\infty}$ is an orbit of Φ then a solution of

Once y is defined on $[0, \tau]$ we can repeat the process.

Let $F\colon \mathbb{R} \to \mathbb{R}$ be a smooth function and consider the class of state dependent delay differential equations given by

$$\dot{y} = F[y(t - \tau + \tau\epsilon y(t))]. \tag{3}$$

Note that if $\epsilon = 0$ we are in the case of a constant delay differential equation where the right hand side depends only on history.

Remark 1 (Linear dissipation)**.** *A small, but interesting modification of Equation* (3) *is as follows. Consider the case of an additional linear term term in the right hand side, of the form*

$$\dot{y} = -ay + F[y(t - \delta(y(t))]. \tag{4}$$

Then ...

In order to illustrate the numerical the numerical procedure we fix two specific examples.

- **The Cubic Ikeda family:** consider

$$\dot{y} = y(t - \delta(y(t)) - y^3(t - \delta(y(t)), \quad \text{i.e.} \quad F(u) = u - u^3.$$

When $\epsilon = 0$ the system was studied in ....LITTERATURE TO BE DONE ....

- **The Mackey-Glass Family:** consider

$$\dot{y} = -ay + \beta \frac{y(t - \delta(y(t))}{1 + y^n(t - \delta(y(t))}, \quad \text{i.e.} \quad F(u) = \beta \frac{u}{1 + u^n}$$

where $\beta,\ n > 0$. When $\epsilon = 0$ this is the example ....LITTERATURE TO BE DONE ...

1.1 Preliminaries

The delay in equation (4) is rescaled in the following manner. Write $x(t) = y(\tau t)$, from (4) we get

$$\dot{x} = \tau \bigg(ax + F[x(t - \frac{1}{\tau}\delta(x(t)))] \bigg). \tag{5}$$

Since $\delta(u)/\tau = 1 - \varepsilon u$, (5) finally writes

$$\dot{x} = \bigg(-a\tau x + \tau F[x(t - 1 + \varepsilon x(t))] \bigg). \tag{6}$$

The gaol of this article is to elaborate an algorithm to compute the solutions of (6). Let us observe that (6) can be seen as a perturbation of the following phase independent delay equation.

$$\dot{x} = \tau \bigg(-ax + F[x(t-1)] \bigg), \qquad t \geq 0 \tag{7}$$

The main part of the theory is exposed in the friction free case, i.e., $a = 0$. At the end of this article we will show that the theory can easily be extending from the friction free case to the general case thanks to an integrating factor. Our first approach consists in understanding the friction-free version of the problem, that is when $\varepsilon = 0$.

1.2 The case of phase independent delay

The general solution of (7) can be seen as the solution of the following Initial Value Problem. Let $\mathbf{C}^0([-1, 0])$ be the set of continuous functions defines on the interval $[-1, 0]$. For a given $z_0 \in \mathbf{C}^0([-1, 0])$, (7) admits a unique solution $z = z(t)$ such that

$$z_0(t) = z(t), \quad t \in [-1, 0].$$

The global solution $z(t)$ is constructed inductively by defining $z(t)$ on the 'next unitary interval' $[0,1]$ by writing

$$z(t) = z(0) + \tau \int_0^t F\Big(z(s-1)\Big)ds = z_0(0) + \tau \int_{-1}^{t-1} F\Big(z_0(s)\Big)ds. \qquad (8)$$

To initiate the induction, we define

$$z_1(t) = z(t+1),\ t \in [-1,0],$$

i.e., the solution shifted by one unit at the source, that is the solution of (7) read on the interval $[0,1]$ but rescaled on the interval $[-1,0]$. From (8), we have

$$z_1(t) = z_0(0) + \tau \int_{-1}^{t} F\Big(z_0(s)\Big)ds. \qquad (9)$$

In like manner, defining the sequence (z_n) of functions

$$z_n : [-1,0] \to \mathbb{R}, \quad t \mapsto z(t+n),$$

that is the solution of (7) read on the interval $[n-1,n]$ but rescaled on the interval $[-1,0]$. we have

$$z_{n+1} = \Psi_0(z_n)$$

where

$$\Psi_0 : \mathcal{F} \to \mathcal{F},\ \mathbf{y} \mapsto \Psi_0(\mathbf{y}),$$

with $\mathcal{F} = \{\mathbf{y} : [-1,0] \to \mathbf{R},\ \mathbf{y} \text{ is } C^0\}$ and where

$$\Psi_0(\mathbf{y})(t) = \mathbf{y}(0) + \tau \int_0^{t+1} F\Big(\mathbf{y}(s-1)\Big)ds = \mathbf{y}(0) + \tau \int_{-1}^{t} F\Big(\mathbf{y}(s)\Big)ds. \qquad (10)$$

Another way to see the above construction is to consider the *stroboscopic map* associated to a known solution of (7), i.e., for each solution $x = \{x(t),\ t \geq -1\}$ of (7), we first consider the restriction of x on $[-1,0]$, i.e. write $x_0(t) = x(t),\ t \in [-1,0]$ and define $x_1 = \Psi(x_0)$ where

$$x_1 : [-1,0] \to \mathbb{R},\ t \mapsto x(t+1),$$

and by induction on n, $x_{n+1} = \Psi_0(x_n)$. In this way, one can see the solution of (7) as the orbit of the stroboscopic map Ψ_0 with a given initial condition $x_0 \in \mathcal{F}$.

2 The state dependent case

Recall that a solution $x = x(t)$ of (6) needs to satisfy

$$\left\{ \begin{array}{rcll} \dfrac{dx}{dt} & = & \tau F\Big(x(t-1+\varepsilon x(t))\Big), & t \geq 0 \\ \\ x(t) & = & x_0(t), & t \in \mathrm{I}_0 \end{array} \right. \qquad (11)$$

where I_0 is choosen sufficiently large, say $\mathrm{I}_0 = [-3/2, 0]$. We first introduce a couple of definitions.

Let $\mathrm{I} \subset \mathbb{R}$ be a compact interval and $B > 0$. We say that a function $g : \mathrm{I} \to \mathbb{R},\ t \mapsto g(t)$ is B-lipschitz continuous if

$$|g(t_1) - g(t_2)| \leq B|t_2 - t_1|, \quad \forall t_2,\ t_1 \in \mathrm{I}.$$

We define $\mathbf{C}^{0,B}(\mathrm{I})$ as the set of B-Lipschitz continuous function defined on I, $\mathbf{C}^{0,B}(\mathrm{I})$ being equipped with the norm $\|\cdot\|_\infty$, i.e.,

$$\|\mathbf{f}\|_\infty = \sup_{t\in \mathrm{I}} |\mathbf{f}(t)|.$$

From now we assume that the initial condition x_0 belongs to $\mathbf{C}^{0,B}(\mathrm{I}_0)$. As a consequence, if ε satisfies $\varepsilon B < 1$, for all $\zeta \in \mathbf{C}^{0,B}(\mathrm{I}_0)$, the map

$$u :\ [0, 1/2] \to \mathbb{R} \quad t \mapsto u(t) = t - 1 + \varepsilon\zeta(t)$$

takes its range in I_0. This initial condition being set, our goal is to construct a solution x of (11) defined for all positive real number such that x coincides with x_0 on I_0. Our approach is to extend the definition of x on a larger domain, by showing that this extension satisfies a fixed point argument.

2.1 A contraction operator

Let $\lambda > 0$, $B > 0$ and $z \in \mathbf{C}^{0,B}([-\lambda, 0])$. We denote by

$$M = \sup_{-\lambda\leq t\leq 0} |z(t)|, \quad K_0 = \sup_{|\xi|\leq M} |F(\xi)|, \quad K_1 = \sup_{|\xi|\leq M} |F'(\xi)|. \tag{12}$$

Let

$$\mathbf{C}_0^{0,B}([0, 1/2]) = \{\mathbf{f} \in \mathbf{C}^{0,B}([0, 1/2]),\ |\ \mathbf{f}(0) = z(0)\},$$

and define the following operator

$$\mathbb{O}_z : \mathbf{C}_0^{0,\ell}([0, 1/2]) \to \mathbf{C}_0^{0,\ell}([0, 1/2]), \quad w \mapsto \mathbb{O}_z(w)$$

where $\ell = \max\{B, \tau K_0\}$ and where

$$\mathbb{O}_z(w)(t) = z(0) + \tau \int_0^t F\Big(z(s - 1 + \varepsilon w(s))\Big) ds. \tag{13}$$

We state the following proposition.

Proposition 1. *Under the above assumptions, there exists $\varepsilon_0 > 0$ such that for all $0 \leq \varepsilon \leq \varepsilon_0$, the operator $\mathbb{O}_z$ is a contraction. More precisely for all $\omega_1,\ \omega_2 \in \mathbf{C}_0^{0,\ell}([0, 1/2])$,*

$$\|\mathbb{O}_z(\omega_1) - \mathbb{O}_z(\omega_2)\|_\infty \leq \frac{1}{2}\|\omega_1 - \omega_2\|_\infty.$$

Observe that $\mathbf{C}_0^{0,\ell}([0,1/2])$ equipped with the norm $\|.\|_\infty$ is a complete space. As a consequence, for each $z \in \mathbf{C}^{0,B}([-\lambda,0])$, $\mathbb{O}_z$ admits a unique fixed point $\Psi(z) \in \mathbf{C}_0^{0,\ell}([0,1/2])$ i.e., satisfies

$$\Psi_\varepsilon(z)(t) = z(0) + \tau \int_0^t F\Big(z(s-1+\varepsilon\Psi_\varepsilon(z)(s))\Big)ds.$$

Moreover, the function

$$z_{1/2} : [-\lambda-1/2, 0] \to \mathbb{R},\ t \mapsto z_{1/2}(t)$$

where

$$\begin{array}{rcll} z_{1/2}(t) & = & z(t+1/2) & \text{if } -\lambda-1/2 \le t \le -1/2, \\ \\ & = & \Psi_\varepsilon(z)(t+1/2) & \text{if } -1/2 \le t \le 0 \end{array}$$

is ℓ-Lipschitz. The map

$$\mathbf{C}^{0,B}([-\lambda,0]) \to \mathbf{C}_0^{0,\ell}([0,1/2]),\ z \mapsto \Psi_\varepsilon(z)$$

is called the Half-Stroboscopic map. Furthermore $\Psi_\varepsilon(z)$ is differentiable on $(0,1/2)$ and we have

$$\frac{d}{dt}\Psi(z)(t) = \tau F\Big(x(t-1+\varepsilon\Psi_\varepsilon(z)(t))\Big).$$

Proof of Proposition 1: We first need to show that the map $\mathbb{O}_z$ takes its ranges in $\mathbf{C}_0^{0,\ell}([0,1/2])$. For all $0 \le t \le 1/2$ and for all $\omega \in \mathbf{C}_0^{0,\ell}([0,1/2]$ we have

$$|\omega(t) - \omega(0)| \le \ell/2, \quad i.e.,\ |\omega(t)| \le |z(0)| + \ell/2.$$

Choose ε_1 small enough such that for all $0 < \varepsilon \le \varepsilon_1$, and for all $0 \le s \le 1/2$,

$$-\lambda \le -1 - \varepsilon_1(|z(0)| + \ell/2) \le -1 - \varepsilon|\omega(s)|$$

and

$$t - 1 + \varepsilon\omega(t) \le -1/2 + \varepsilon_1(|z(0)| + \ell/2) \le 0.$$

and thus for all $s \in [0,1/2]$ and for all $w \in \mathbf{C}_0^{0,\ell}([0,1/2])$,

$$-\lambda \le s - 1 + \varepsilon w(s) \le 0.$$

The operator is thus well defined. Let $0 \le t_1 \le t_2 \le 1/2$. Observe that $\mathbb{O}_z(w)(0) = z(0)$ and

$$\begin{array}{rcl} |\mathbb{O}_z(w)(t_2) - \mathbb{O}_z(w)(t_1)| & \le & \tau \displaystyle\int_{t_1}^{t_2} \Big|F\Big(z(s-1+\varepsilon w(s))\Big)\Big| ds \\ & \le & \tau K_0(t_2 - t_1) \le \ell(t_2 - t_1), \end{array}$$

meaning that $\mathbb{O}_z$ leaves $\mathbf{C}_0^{0,\ell}([0,1/2])$ invariant. Let $w_1, w_2 \in \mathbf{C}_0^{0,\ell}([0,1/2])$. We have

$$\mathbb{O}_x(w_1)(t) - \mathbb{O}_x(w_2)(t) = \tau \int_0^t \Big(\Delta(w_1,w_2)(s)\Big) ds$$

where

$$\Delta(w_1,w_2)(s) = F\Big(z(s-1+\varepsilon w_1(s))\Big) - F\Big(z(s-1+\varepsilon w_2(s))\Big).$$

Thanks to the Mean Value Theorem, we have

$$\Big\|\Delta(w_1,w_2)\Big\|_\infty \leq \varepsilon B \sup_{|\xi|\leq M} |F'(\xi)| \cdot \|w_1 - w_2\|_\infty.$$

Therefore, it follows that

$$\|\mathbb{O}_z(w_1) - \mathbb{O}_z(w_2)\|_\infty \;\leq\; \varepsilon\tau \frac{K_1}{2} B \|w_1 - w_2\|_\infty$$

and therefore, by choosing

$$0 < \varepsilon_0 \leq \min\{\varepsilon_1, \frac{1}{\tau B K_1}\},$$

the proof of Proposition 1 is completed.

2.2 The half Stroboscopic map

We are now in position to construct the solution x of (11) with initial condition x_0 on each interval of the form

$$[m/2, (m+1)/2], \quad \text{where} \quad m = 0, 1, \dots.$$

In what follows we use the following notation

$$x_{m/2} \;:\; [-1+\varepsilon x_0(0) - m/2, 0] \to \mathbb{R}, \quad t \mapsto x(t+m/2), \quad m = 0,1,\dots,$$

$$y_{m/2} \;:\; [0,1/2] \to \mathbb{R}, \qquad t \mapsto x(t+m/2), \qquad m = 0,1,\dots.$$

Our first search will be for a function $y_0 : [0,1/2] \to \mathbb{R}$ such that

$$\dot{y}_0(t) = \tau F\Big(x_0(t-1+\varepsilon y_0(t))\Big). \tag{14}$$

i.e., satisfies

$$y_0(t) = \mathbb{O}_{x_0}(y_0)(t) = x_0(0) + \tau \int_0^t F\Big(x_0(s-1+\varepsilon y_0(s))\Big) ds, \tag{15}$$

i.e., y_0 is the fixed point of $\mathbb{O}_{x_0}$ guaranteed by Proposition 1 (with $\lambda = 1 - \varepsilon x_0(0)$). The solution of (11) is now known on $[-1+\varepsilon x_0(0), 1/2]$ and we write

$$x(t) = \begin{cases} x_0(t) & \text{if} \quad -1+\varepsilon x_0(0) \le t \le 0, \\ \\ y_0(t) & \text{if} \quad 0 \le t \le 1/2, \end{cases}$$

or simply (using the above notation)

$$x(t) = x_{1/2}(t-1/2), \quad -1+\varepsilon x_0(0) \le t \le 1/2,$$

which satisfies (11) for all $0 < t \le 1/2$. The next step consists of extending the solution x on the intervals

$$[1/2, 1] \cup [1, 3/2] \cup \cdots .$$

Assume the $y_{j/2}$'s, $j = 0, \ldots, m-1$ to be known. Observe that this also implies that the $x_{j/2}$'s, $j = 1, \ldots, m$ are known. From (11) and the notation above, we have

$$\dot{y}_{m/2} = \tau F(x_{m/2}(t - 1 + \varepsilon(y_{m/2}(t))),$$

equivalently for $0 \le t \le 1/2$

$$\begin{aligned} y_{m/2}(t) &= \mathbb{O}_{x_{m/2}}(y_{m/2}) \\ &= x_{m/2}(0) + \tau \int_0^t F\Big(x_{m/2}(s - 1 + \varepsilon(y_{m/2}(s))\Big) ds, \end{aligned}$$

i.e., $y_{m/2}$ is the fixed point of $\mathbb{O}_{x_{m/2}}$ and since $x_{m/2}$ is Lipchitz, the existence and uniqueness of $y_{m/2}$ is again guaranteed by Proposition 1 (with $\lambda = 1 - \varepsilon x_0(0) + m/2$). In other words we have

$$y_{m/2} = \Psi_\varepsilon(x_{m/2}), \quad m \ge 0.$$

The solution of (11) is now known on $[-1+\varepsilon x_0(0), (m+1)/2)]$ and we write

$$x(t) = \begin{cases} x_{m/2}(t - m/2) & \text{if} \quad -1+\varepsilon x_0(0) \le t \le m/2, \\ \\ y_{m/2}(t - m/2) & \text{if} \quad m/2 \le t \le (m+1)/2, \end{cases}$$

or simply

$$x(t) = x_{(m+1)/2}(t - (m+1)/2) \quad \text{if } t \le (m+1)/2,$$

which satisfies (11) for all $0 < t \le (m+1)/2$.

Remark: The solution x retrieved that way is Lipschitz everywhere but the Lipschitz constant may increase after each iteration of the Half Stroboscopic map and therefore one may have to choose ε_0 smaller and smaller. However, even if x_0 is not differentiable, one easily see that x is differentiable for all $t > 0$ and the degree of differentiability increases as t increases. This implies that if x is bounded with bounded derivatives, those Lipschitz constant can *a-posteriori* be chosen uniformly bounded.

2.3 Adding friction

We now are in position to understand the dynamics of the full system (6) when the friction is taken into consideration. Recall that the system to be considered satisfies

$$\dot{x} = -a\tau x + \tau F[x(t-1+\varepsilon x(t))], \tag{16}$$

with initial condition $x(t) = x_0(t)$ defined on $[-1+\varepsilon x_0(0), 0]$. We adapt our previous approach by using the method of integrating factor. From (16) we write

$$\frac{d}{dt}\Big(x(t)e^{a\tau t}\Big) = \tau e^{a\tau t} F[x(t-1+\varepsilon x(t))].$$

By integrating both sides of the former equation we get

$$x(t) = x(0)e^{-a\tau t} + \tau \int_0^t e^{a\tau(s-t)} F[x(s-1+\varepsilon x(s))]ds. \tag{17}$$

To this point our strategy will follow the same as in section 2.1. In what follows, the notation used are the same as before. Let $\lambda > 0$, $B > 0$ and $z \in \mathbf{C}^{0,B}([-\lambda, 0])$. We define the following operator

$$\mathbb{J}_z : \mathbf{C}^{0,\ell}[(0,1/2)] \to \mathbf{C}^{0,\ell}[(0,1/2)],\ \omega \mapsto \mathbb{J}_z(\omega)$$

where $\ell = \max\{B, \tau K_0 + z(0)a\tau\}$ and where

$$\mathbb{J}_z(\omega)(t) = z(0)e^{-a\tau t} + \tau \int_0^t e^{a\tau(s-t)} F[z(s-1+\varepsilon\omega(s))]ds.$$

We state the following proposition.

Proposition 2. *There exists $\varepsilon_0 > 0$ such that for all $0 \leq \varepsilon \leq \varepsilon_0$, the operator $\mathbb{J}_z$ is a contraction. More precisely for all $\omega_1,\ \omega_2 \in \mathbf{C}_0^{0,\ell}([0,1/2])$,*

$$\|\mathbb{J}_z(\omega_1) - \mathbb{J}_z(\omega_2)\|_\infty \leq \frac{1}{2}\|\omega_1 - \omega_2\|_\infty.$$

The proof of Proposition 2 is similar to that of Proposition 1 and is left to the reader. Observe however, in this case, the constant ℓ has to be chosen a bit bigger than in the case of Proposition 1. More precisely, we easily verify that since $a > 0$, the term $z(0)e^{-a\tau t}$ is $z(0)a\tau$-Lipshitz since $a > 0$ and therefore for all bounded function ω, $\mathbb{J}_z(\omega)$ is $\tau K_0 + z(0)a\tau$-Lipshitz. The consequence of Proposition 2, is that we can construct the Half-Stroboscopic map

$$\Psi_\varepsilon : \mathbf{C}^{0,B}([-\lambda, 0]) \to \mathbf{C}^{0,\ell}([0,1/2]), \quad z \mapsto \Psi_\varepsilon(z)$$

where $\Psi_\varepsilon(z)$ is the unique fixed point of $\mathbb{J}_z$, i.e.,

$$\Psi_\varepsilon(z)(t) = z(0)e^{-a\tau t} + \tau \int_0^t e^{a\tau(s-t)} F\Big(z(s-1+\varepsilon\Psi_\varepsilon(z)(s))\Big)ds,$$

and the same construction as in Section 2.2 is deduced.

3 A numerical method

One of the difficulty we have to face when studying delay differential equation is that we are working in an infinite dimensional space. In particular our techniques relies on the existence of a fixed point in an infinite dimensional space. To overcome this difficulty, we replace the contracting operator by an interpolating operator. The latter is still a contraction and is arbitrarily close to the former in the C^0 topology. The method we are describing now is presented, for sake of simplicity, in the case friction free case (i.e., $a = 0$), but this method is also valid in the case the system admit friction.

3.1 The Lagrange-Chebychev interpolation

Let $q > 1$ be an integer. We denote by $\mathbb{P}_q[t]$ the subset of polynomial functions of degree less than $q-1$. We define the Lagrange Chebyschev interpolating operator

$$\mathcal{L}_q : \mathbf{C}^0([0,1/2]) \to \mathbb{P}_q[t],\ h \mapsto \mathcal{L}_q(h)$$

where

$$\mathcal{L}_q(h)(t) = P_q(\hat{h})(4t-1), \quad t \in [0,1/2], \quad P_q(\hat{h})(u) = \sum_{j=0}^{q-1} c_j T_j(u),$$

where

$$\hat{h}(u) = h((u+1)/4), \quad -1 \le u \le 1,$$

the T_j's being the Chebyshev polynomial i.e.,

$$T_j(u) = \cos(j \arccos(u)),\ j = 0, \dots, q-1, \quad u \in [-1,1],$$

$$c_j = \frac{2}{q} \sum_{k=0}^{q-1} \hat{h}(u_k) T_j(u_k), \quad j > 0, \quad c_0 = \frac{1}{q} \sum_{k=0}^{q-1} \hat{h}(u_k),$$

where the u_k's are the Chebyshev node on $[-1,1]$, i.e.,

$$u_k = \cos(\frac{2k+1}{2q}\pi), \quad k = 0, \dots, q-1.$$

See [**?**] for more details.

The operator $\mathcal{L}_q$ is linear, and for all continuous function, $\mathcal{L}_q(h)$ converges uniformly to h on $[0,1/2]$ as q tends to ∞. More precisely we can state the following lemma.

Lemma 1. *Let $B > 0$ and let $z \in \mathbf{C}^{0,B}([0,1/2])$. Then*

$$\|\mathcal{L}_q(z) - z\|_\infty \le \frac{(1+\mu_q)}{4q} B$$

where

$$\mu_q = \frac{1}{\pi}\sum_{j=0}^{q-1} \cot\left(\frac{(j+1/2)\pi}{2q}\right) = \frac{2}{\pi}\log(q) + 0.9625 + \mathcal{O}(1/q)$$

REF MH6 to be added: J.C. Mason, David C. Handscomb, Chebyshev Polynomials, CRC Press, Sep 17, 2002.

This lemma is a direct consequence of Jackson's Theorem and its Corollary 6.14A in [**?**]. Both results are formulated for a continuous function defined on $[-1,1]$ with modulus of continuity

$$\mathbf{m}(\delta) = \sup_{|x_1-x_2|\le\delta} |h(x_1) - h(x_2)|.$$

Corollary 6.14A in [**?**] states that

$$\|P_q(\hat{z}) - \hat{z}\|_\infty \le \mathbf{m}(1/q)(1+\mu_q)$$

After a linear rescaling one extends these results for interpolation on the interval $[0,1/2]$ and in the present case, since z is B-Lipschiz on $[0,1/2]$, one easily see that

$$\hat{z}(u) = z((u+1)/4)$$

is $(B/4)$-Lipschitz on $[-1,1]$ and therefore

$$\mathbf{m}(1/q) \le \frac{B}{4q},$$

and the lemma follows. Finally, observe that the estimate

$$\mu_q = \frac{2}{\pi}\log(q) + 0.9625 + \mathcal{O}(1/q)$$

is also given in [**?**], which implies that

$$\|\mathcal{L}_q(z) - z\|_\infty \to 0 \quad \text{as } q \to \infty$$

independently from the choice of z.

Corollary 1. *Under the same assumptions as in the above lemma, then there exists $q_0 > 1$ such that for all $q \ge q_0$,*

$$\|\mathcal{L}_q(z)\|_\infty \le 2\|z\|_\infty.$$

Proof: Since $\mathcal{L}_q$ is linear it is sufficient to show the lemma in the case $\|z\|_\infty = 1$. From the above lemma, choose $q_0 > 0$ sufficiently large such that for all $q > q_0$

$$(1+\mu_q)B/(4q) \le 1.$$

We have

$$\|\mathcal{L}_q(z)\|_\infty \le \|\mathcal{L}_q(z) - z\|_\infty + \|z\|_\infty \le 2,$$

ending the proof of the corollary.

3.2 The reduced operator

The limitation we have with the contraction operator introduced in the former section is that we have to work in infinite dimension space. To overcome this difficulty we replace the contraction operator by the so called *reduced operator*. The latter is an approximation of the former. We show that the reduced operator is also a contraction.

Let $\lambda > 1$, $B > 0$ and $z \in \mathbf{C}^{0,B}([-\lambda, 0])$. We define

$$\mathbb{O}_{z,q} : \mathbb{P}_q[t] \cap \mathbf{C}_0^{0,\ell}([0,1/2]) \to \mathbb{P}_q[t] \cap \mathbf{C}_0^{0,\ell}([0,1/2]),\ \mathbf{f} \mapsto \mathbb{O}_{z,q}(\mathbf{f})$$

where $\ell = \max\{2\tau K_0, B\}$ and where

$$\mathbb{O}_{z,q}(\mathbf{f})(t) \quad = \quad z(0) + \tau \int_0^t \mathcal{L}_{q-1}\Big(F(z(s-1+\varepsilon \mathbf{f}(s)))\Big) ds. \qquad (18)$$

We state the following proposition.

Proposition 3. *There exists $q_0 \geq 1$ and $\varepsilon_2 > 0$ such that for all $0 \leq \varepsilon \leq \varepsilon_2$, and for all $q > q_0 + 1$, $\mathbb{O}_{z,q}$ is a contraction. More precisely for all $\omega_1,\ \omega_2 \in \mathbb{P}_{q,0}[t] \cap \mathbf{C}^{0,\ell}([-\lambda, 0])$,*

$$\|\mathbb{O}_{z,q}(\omega_1) - \mathbb{O}_{z,q}(\omega_2)\|_\infty \leq \frac{1}{2}\|\omega_1 - \omega_2\|_\infty.$$

Proof: We first verify that $\mathbb{O}_{z,q}$ leaves $\mathbb{P}_q[t] \cap \mathbf{C}_0^{0,\ell}([0,1/2])$ invariant. For all $0 \leq t \leq 1/2$ and for all $\omega \in \mathbf{C}_0^{0,\ell}([0,1/2]$ from its definition $\mathbb{O}_{z,q}(w)$ is a polynomial of degree equal or less than $q-1$ and $\mathbb{O}_{z,q}(w)(0) = z(0)$. Furthermore we have

$$|\omega(t) - \omega(0)| \leq \ell/2, \quad i.e.,\ |\omega(t)| \leq |z(0)| + \ell/2.$$

Choose ε_1 small enough such that for all $0 < \varepsilon \leq \varepsilon_1$, and for all $0 \leq s \leq 1/2$,

$$-\lambda \leq -1 - \varepsilon_1(|z(0)| + \ell/2) \leq -1 - \varepsilon|\omega(s)|$$

and

$$t - 1 + \varepsilon\omega(t) \leq -1/2 + \varepsilon_1(|z(0)| + \ell/2) \leq 0.$$

and thus for all $s \in [0,1/2]$ and for all $w \in \mathbf{C}_0^{0,\ell}([0,1/2])$,

$$-\lambda \leq s - 1 + \varepsilon w(s) \leq 0.$$

The operator is thus well defined. We now verify that $\mathbb{O}_{z,q}(w) \in \mathbf{C}_0^{0,\ell}([0,1/2])$. Let $0 \leq t_1 \leq t_2 \leq 1/2$ and take $q - 1 > q_0$ where q_0 is given in Corollary 1. We have

$$\begin{aligned}|\mathbb{O}_{z,q}(w)(t_2) - \mathbb{O}_{z,q}(w)(t_1)| \quad &\leq \quad \tau \int_{t_1}^{t_2} \left| \mathcal{L}_{q-1}\Big(F\Big(z(s-1+\varepsilon w(s))\Big)\Big) \right| ds \\ &\leq \quad 2\tau K_0(t_2 - t_1) \leq \ell(t_2 - t_1),\end{aligned}$$

meaning that $\mathbb{O}_z$ leaves $\mathbf{C}_0^{0,\ell}([0,1/2])$ invariant.

Let $w_1, w_2 \in \mathbf{C}_0^{0,\ell}([0,1/2])$. A straightforward computation gives

$$\mathbb{O}_{z,q}(w_1)(t) - \mathbb{O}_{z,q}(w_2)(t) = \tau \int_0^t \Big(\Delta_q(w_1,w_2)(s) \Big) ds$$

where

$$\begin{aligned} \Delta_q(w_1,w_2)(s) &= \mathcal{L}_{q-1}\Bigg(F\Big(z(s-1+\varepsilon w_1(s)) \Big) \Bigg) \\ &- \mathcal{L}_{q-1}\Bigg(F\Big(z(s-1+\varepsilon w_2(s)) \Big) \Bigg). \end{aligned}$$

Thanks to the Mean Value Theorem and Corollary 1, we have

$$\Big\| \Delta_q(w_1,w_2) \Big\|_\infty \le 2\varepsilon B \sup_{|\xi|\le M} |F'(\xi)| \cdot \|w_1 - w_2\|_\infty.$$

Therefore, it follows that

$$\|\mathbb{O}_{z,q}(w_1) - \mathbb{O}_{z,q}(w_2)\| \;\le\; \varepsilon\tau K_1 B \|w_1 - w_2\|$$

and therefore, by choosing

$$0 < \varepsilon \le \min\{\varepsilon_1, \frac{1}{2\tau B K_1}\},$$

the proof of Proposition 3 is completed. As a consequence, for each $z \in \mathbf{C}^{0,B}([-\lambda,0])$, $\mathbb{O}_{z,q}$ admits a unique fixed point. Therefore one can deduce the same construction as in section 2.2 replacing $\mathbb{O}_x$ by $\mathbb{O}_{x,q}$.

3.3 Constructing the orbit

Fix ν a small positive number representing the tolerance of our computation. Take $B > 0$ and $x_0 \in \mathbf{C}^{0,B}[-\lambda,0]$ where $\lambda = 1 - \varepsilon x_0(0)$. Our goal is now to compute $\Psi_\varepsilon(x_0)$ with an arbitrary accuracy, more precisely we aim to compute $y_0 = \Psi_\varepsilon(x_0)$ and more precisely to find a function $\tilde{y}_0$ such that

$$\|\Psi_\varepsilon(x_0) - \tilde{y}_0\| \le \nu. \tag{19}$$

Let $q > 1$ be an integer such that for all $\omega \in \mathbf{C}^{0,B}([0,1/2])$,

$$\|\mathcal{L}_{q-1} G_\omega(t) - G_\omega(t)\|_\infty \le \nu/(2\tau),$$

where

$$G_\omega(t) = F\Big(x_0(t-1+\varepsilon\omega(t)) \Big).$$

The above implies that for all $\omega \in \mathbf{C}^{0,B}([0,1/2])$ we have

$$\|\mathbb{O}_{x_0}(\omega) - \mathbb{O}_{x_0,q}(\omega)\|_\infty \leq \nu/4. \tag{20}$$

The existence of such integer q is guaranteed by Lemma 1. Let $\mathbf{f}_0 \in \mathbf{C}^0([0,1/2])$ and construct the sequence of functions $(\mathbf{f}_n)$ in $\mathbf{C}^0([0,1/2])$ such that

$$\mathbf{f}_{n+1}(t) = \mathbb{O}_{x_0,q}(\mathbf{f}_n)(t).$$

Thanks to Proposition 3, there exists an integer $\mathbf{m} \geq 1$ such that

$$\|\mathbb{O}_{x_0,q}(\mathbf{f}_n) - \mathbf{f}_n)\| \leq \nu/4, \quad \forall n \geq \mathbf{m}. \tag{21}$$

We now write $\tilde{y}_0 = \mathbf{f}_\mathbf{m}$. By definition we have

$$\mathbb{O}_{x_0}(\Psi_\varepsilon(x_0)) = \Psi_\varepsilon(x_0)$$

and we also have

$$\begin{aligned} \|\Psi_\varepsilon(x_0) - \mathbf{f}_\mathbf{m}\|_\infty &= \|\mathbb{O}_{x_0}(\Psi_\varepsilon(x_0)) - \mathbf{f}_\mathbf{m}\|_\infty \\ &\leq \|\mathbb{O}_{x_0}(\Psi_\varepsilon(x_0)) - \mathbb{O}_{x_0}(\mathbf{f}_\mathbf{m})\|_\infty \\ &+ \|\mathbb{O}_{x_0}(\mathbf{f}_\mathbf{m}) - \mathbb{O}_{x_0,q}(\mathbf{f}_\mathbf{m})\|_\infty + \|\mathbb{O}_{x_0,q}(\mathbf{f}_\mathbf{m}) - \mathbf{f}_\mathbf{m}\|_\infty \end{aligned} \tag{22}$$

From Proposition 1 we have

$$\mathbb{O}_{x_0}(\Psi_\varepsilon(x_0)) - \mathbb{O}_{x_0}(\mathbf{f}_\mathbf{m})\|_\infty \leq \frac{1}{2}\|\Psi_\varepsilon(x_0) - \mathbf{f}_\mathbf{m}\|_\infty,$$

furthermore from (20) we have

$$\|\mathbb{O}_{x_0}(\mathbf{f}_\mathbf{m}) - \mathbb{O}_{x_0,q}(\mathbf{f}_\mathbf{m})\|_\infty \leq \nu/4$$

which implies together with (21)

$$\frac{1}{2}\|\mathbb{O}_{x_0}(\Psi_\varepsilon(x_0)) - \mathbf{f}_\mathbf{m}\|_\infty \quad \leq \quad \nu/4 + \nu/4, \tag{23}$$

and finally

$$\|\mathbb{O}_{x_0}(\Psi_\varepsilon(x_0)) - \tilde{y}_0\|_\infty \leq \nu.$$

This above procedure allows us to construct $y_0 \sim \tilde{y}_0$, we then deduce

$$x_{1/2} : [-1 + \varepsilon x_0(0) - 1/2, 0] \to \mathbb{R},\ t \mapsto x(t + 1/2)$$

where

$$x(t) = x_0(t) \quad \text{if } t < 0, \quad x(t) = \tilde{y}_0(t) \quad \text{if } 0 \leq t \leq 1/2.$$

Following the above construction we deduce $\tilde{y}_{1/2}$ which approximate $y_{1/2}$ up to a ν-tolerance and following the same notation as in section 2.2, we can retrieve the solution x on the entire real line.

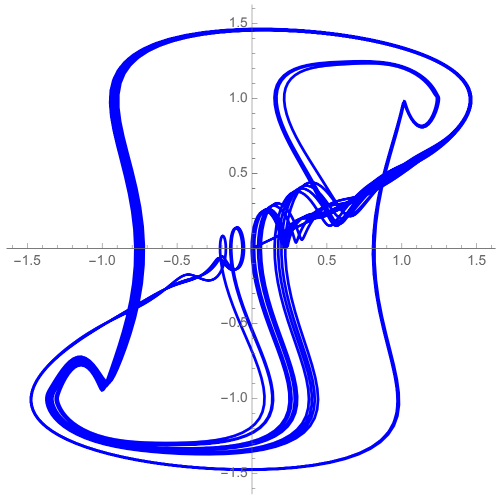

Figure 1: Lissajou phase portrait for $\varepsilon = 1/100$, $a = 0$, $\tau = 1.59$

We first illustrate our technique by displaying the Lissajou phase portrait of the cubic family for different value of the parameters. By Lissajou we mean to plot the parametric curve

$$\{(x(t), x(t-1)), \ | \ 0 \leq t \leq 150.\}$$

Since we rescaled the time by a factor τ this indeed represents

$$\{y(t), y(t-\tau)), \ | \ 0 \leq t \leq 150\tau\}.$$

We also illustrate our technique for the MacKey glass family and display several Lissajou phase portraits for different value of the parameters.

4 Numerical Results

5 Experiments in dimension

References

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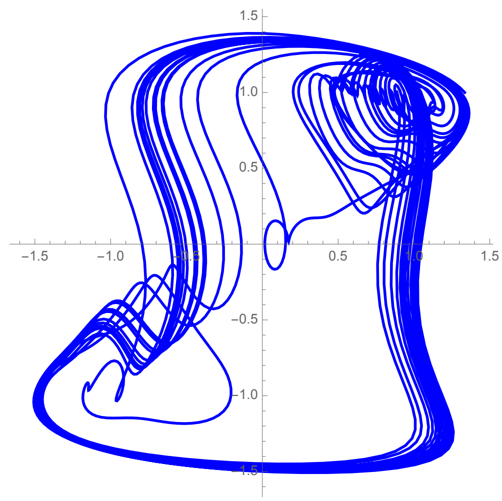

Figure 2: Lissajou phase portrait for $\varepsilon = 1/10$, $a = 0.06734$, $\tau = 1.98$

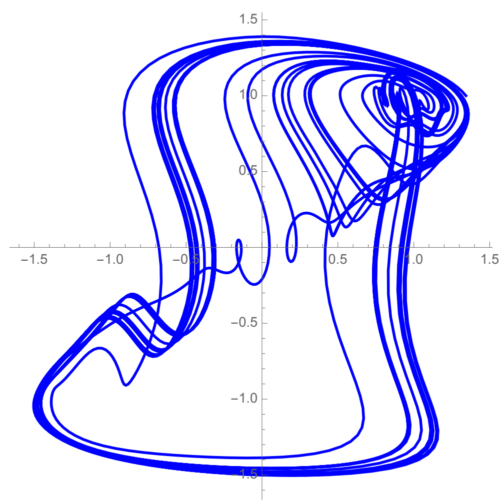

Figure 3: Lissajou phase portrait for $\varepsilon = 1/10$, $a = 0.0563$, $\tau = 1.87$

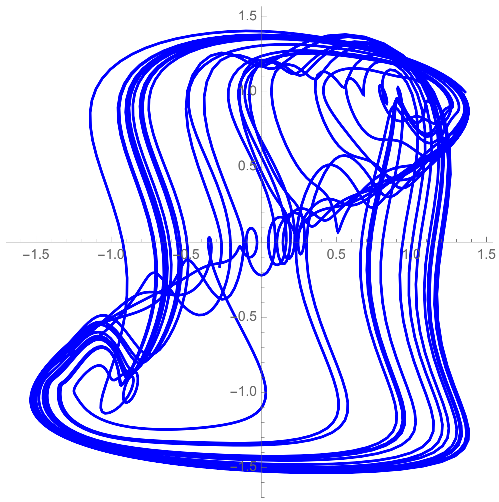

Figure 4: Lissajou phase portrait for $\varepsilon = 1/10$, $a = 0.05425$, $\tau = 1.94$

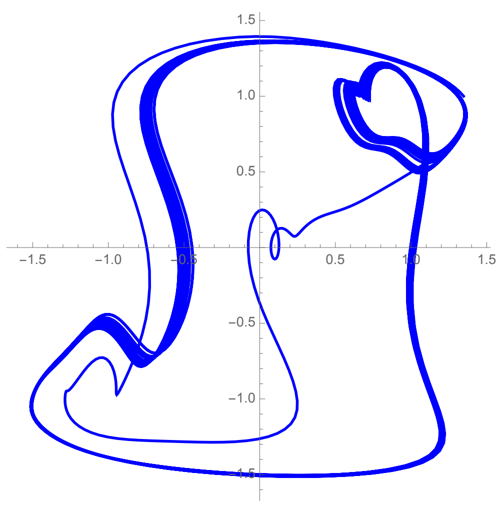

Figure 5: Lissajou phase portrait for $\varepsilon = 1/10$, $a = 0.0557$, $\tau = 1.89$

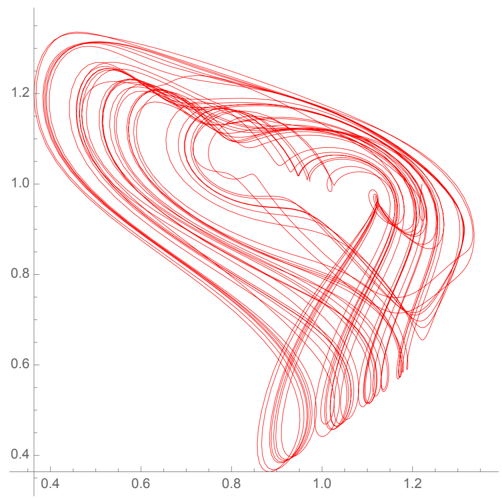

Figure 6: Lissajou phase portrait for $\tau = 2$, $\varepsilon = 0.086$, $\gamma = 1$, $n = 9.65$. This figures mimics the well known Lissajou phase portrait for this family

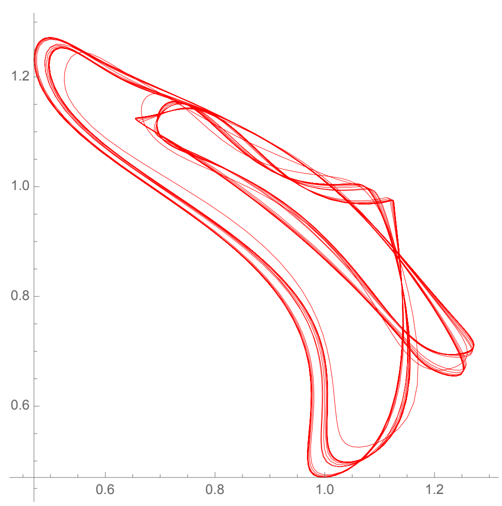

Figure 7: Lissajou phase portrait for $\tau = 2$, $\varepsilon = 0.22$, $\gamma = 1$, $n = 9.65$. This figures mimics the well known Lissajou phase portrait for this family

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