Analysis Qualifier
Fall 2023

INSTRUCTIONS. Write clearly and indicate clearly where your answer to each exercise begins and ends.

Write on only one side of the paper. Anything written on the reverse side will be ignored.

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1. (a) Assume $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous and has the property that the inverse image of every bounded set is bounded. Prove: f assumes a maximum value or f assumes a minimum value.

   Hints: For some $a > 0$ the set $f^{-1}([-a, a])$ is compact non-empty. As such it is contained in some closed disc D around the origin. You may use without proof, and probably should use, that $\mathbb{R}^2 \backslash D$ is connected.

   (b) Prove that the result is false for continuous $f : \mathbb{R} \to \mathbb{R}$.

2. Let $\{x_n\}$ be a sequence in a metric space M converging to a point $L \in M$. Prove that $\{x_n : n \geq 1\} \cup \{L\}$ is a compact **set**.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and let
$$f_n(x) = \frac{1}{n} \sum_{k=0}^{n-1} f\left(x + \frac{k}{n}\right).$$
   Prove that (f_n) converges uniformly to a limit on every interval $[a, b]$.

4. Brouwer's celebrated *theorem of invariance of domain* states that if $U$is open in $\mathbb{R}^n$ and $f : U \to \mathbb{R}^n$ is injective and continuous, then $f(U)$ is an open subset of $\mathbb{R}^n$. Use this theorem to prove: If $n, m \in \mathbb{N}$ and $n \neq m$ then $\mathbb{R}^n$ and $\mathbb{R}^m$ are **not** homeomorphic.

5. Compute
$$\lim_{n\to\infty} \int_1^n \frac{n[\log(n + x) - \log n]}{x^4}\, dx$$
   Be sure to justify all steps.

6. Assume $f : [0, 1] \to \mathbb{R}$ is continuous. Prove that the graph of f; the set
$$G(f) = \{(x, f(x)) : 0 \leq x \leq 1\},$$
   is a subset of $\mathbb{R}^2$ of measure zero.

7. For each $n \in \mathbb{N}$ assume $f_n : [0, 1] \to [0, 1]$ and assume there exists a constant $M \geq 0$ such that $|f_n(x) - f_n(y)| \leq M|x - y|$ for all $x, y \in [0, 1]$ and $n \in \mathbb{N}$. Assume also that $\lim_{n\to\infty} f_n(x)$ exists for all $x \in [0, 1]$. Defining $f : [0, 1] \to \mathbb{R}$ by
$$f(x) = \lim_{n\to\infty} f_n(x)$$
   for $x \in [0, 1]$, prove that the sequence $\{f_n\}$ converges uniformly to f.

Analysis Qualifying Exam
Spring 2023

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1. Let (X, d) be a compact metric space. Show that for every $\varepsilon > 0$, there exists a finite collection of ε-balls whose union contains X (where an ε-ball is a set of the form $\{x \in X : \ d(x, c) < \varepsilon\}$ for some $c \in \ X$.)

2. Let (x_n) be a sequence in $\mathbb{R}$ such that $\sum_{n=1}^{\infty} |x_n - x_{n-1}|$ is convergent. Show that (x_n) is a Cauchy sequence.

3. Let $I = [a, b]$, and let $f : I \to \mathbb{R}$ be continuous. Assume that for each $x \in I$, there is a $y \in I$ such that $|f(y)| \leq \frac{1}{2}|f(x)|$. Show that there is a point $c \in I$ such that $f(c) = 0$.

4. Let (f_n) be a sequence of a continuous functions which converges uniformly to a function f on a set $E \subseteq \mathbb{R}$. Prove that
$$\lim_{n\to\infty} f_n(x_n) = f(x)$$
for every sequence (x_n) in E that converges to x.

5. Let K be a compact subset of $\mathbb{R}$, and let $\{f_n\}_{n\geq 1}$ be a uniformly bounded and equicontinuous family of functions $K \to \mathbb{R}$. For each $n \geq 1$, define
$$g_n(x) = \max\{f_1(x), \ldots, f_n(x)\}.$$
   (a) Show that the family $\{g_n\}_{n\geq 1}$ is equicontinuous on K.
   (b) Show that the sequence (g_n) is increasing, that is, $g_n(x) \leq g_{n+1}(x)$ for all $x \in K$, and all $n \geq 1$.
   (c) Show that the sequence (g_n) converges uniformly on K.

6. Suppose $f_n \geq 0$ for all $n \geq 1$, $f_n \to f$ a.e. on $[0, \infty)$, and there exists $M > 0$ such that
$$\liminf_{n\to\infty} \int_E f_n \, dm \leq M m(E)$$
for every measurable set $E \subseteq \mathbb{R}$, where m denotes the Lebesgue measure. Show that
$$m\left(\{x \in [0, \infty) : \ f(x) > M\}\right) = 0.$$

7. Suppose that $f : [0, 1]$ is continuous. Prove that
$$\lim_{n\to\infty} \int_0^1 f(x^n) \, dx$$
exists and find the limit. Does the limit always exist if f is only assumed to be Lebesgue integrable?

Analysis Qualifying Exam
Fall 2022

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1. Let $a_n \geq 0$ for all $n \geq 1$ and assume that $\sum_{n=1}^{\infty} a_n$ is convergent. Show that $\sum_{n=1}^{\infty} a_n^2$ converges. Is the result still true if the assumption that $a_n \geq 0$ is dropped?

2. Let $f : (a, b) \to \mathbb{R}$ be continuous, increasing, and bounded. Show that f is uniformly continuous on (a, b).

3. Let (X, d_X) be a connected metric space. Show that if X contains at least two distinct points, then X is uncountable.

4. Consider the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^x}$.

   (a) Find the values of x for the series to be convergent.

   (b) Find the intervals for the series to converge absolutely.

   (c) Find the intervals for the series to converge uniformly.

   (d) Find the intervals for the limit function of the series to be continuous.

5. Let $\mathcal{F}$ be an equicontinuous and bounded set of functions from $\mathbb{R}$ to $\mathbb{R}$, and let $g(x) = \sup\{f(x) : \ f \in \mathcal{F}\}$. Show that g is continuous.

6. Let (f_n) be a sequence of measurable functions defined on a measurable set $E \subseteq \mathbb{R}$. Assume that

$$\sum_{n=1}^{\infty} m\left(\left\{x \in E : \ |f_n(x)| \geq \frac{1}{n}\right\}\right) < \infty,$$

   where m denotes the Lebesgue measure. Show that $f_n(x) \to 0$ almost everywhere on E.

7. Determine the limit:

$$\lim_{n\to\infty} \int_0^1 \frac{ne^{-x}}{1 + n^2x^2}\, dx.$$

Qualifier Problems-Spring 2022

[1] Let $\{a_n\}$ be a bounded sequence of real numbers and define

$S = \{s \in \mathbb{R}$: there exists a subsequence $\{a_{n_k}\}$ of $\{a_n\}$ such that $\lim_{k\to\infty} a_{n_k} = s\}$.

Prove S is compact.

[2] Consider the series

$$\sum_{n=1}^{\infty} \left(\frac{x^n}{4^n+1} + \frac{n^2}{x^n} \right)$$

1. Show that the series (defined above) converges uniformly in every interval $[1/4+\epsilon, 4-\epsilon]$ for $0 < \epsilon < 15/8$. Does the series also converge uniformly in $(1/4, 4)$?

2. Defining $f : (1/4, 4) \to \mathbb{R}$ by $f(x) = \sum_{n=1}^{\infty} \left(\frac{x^n}{4^n+1} + \frac{n^2}{x^n} \right)$, prove f is continuous in (a, b).

[3] Let $f : \mathbb{R} \to (0, \infty)$ be differentiable everywhere. Assume that

$$\lim_{t\to\infty} f(t) = L. \quad \text{Show that}$$

there exists a sequence (t_n) with $\lim_{n\to\infty} t_n = \infty$ such that $\lim_{n\to\infty} f'(t_n) = 0$.

[4] Compute

$$\lim_{n\to\infty} n \int_0^{\pi} \frac{\cos(x)}{1+n^2x^2}\, dx.$$

[5] Assume $f : [0, \infty) \to \mathbb{R}$ is continuous and integrable. Show that f is uniformly continuous if and only if

$$\lim_{x\to\infty} f(x) = 0.$$

[6] Assume $f \geq 0$ and measurable. Let

$$E_k = \{x \ : \ f(x) > 2^k\}.$$

We assume that f is finite almost everywhere. Show that f is Lebesgue integrable if and only if

$$\sum_{k=-\infty}^{\infty} 2^k \lambda(E_k) < \infty.$$

[7] Suppose f is integrable on $[0, b]$ and

$$g(x) = \int_x^b \frac{f(t)}{t} dt.$$

Show that g is integrable and

$$\int_0^b g(x)dx = \int_0^b f(t)dt$$

Analysis-Qualifying Exam - Fall 2021

[1] Consider the sequence (f_n), where

$$f_n(x) = \frac{e^{-nx}}{1+x^n}.$$

1. Show that the sequence (f_n) converges point wise on $[0,1]$, and find the limit function.
2. Determine if the convergency is uniform.

[2] Show that the function

$$f : \mathbb{R} \to \mathbb{R},\ x \mapsto \sqrt{|x|}$$

is uniformly continuous.

[3] Let a_n be a non negative sequence. Show that

$$\sum_{n\geq 1} a_n \quad \text{converges} \Rightarrow \sum_{n\geq 1} \sqrt{a_n}/n \quad \text{converges.}$$

Is the converse true?

[4] Show that there exists an injection $L : \mathbb{N} \to \mathbb{N}$ such that

$$\sum_{j=1}^{\infty} (-1)^{L(j)}/L(j) = 15.$$

[5] Compute

$$\lim_{n\to\infty} \int_0^n \frac{n \sin(u)}{u(1+n^2u^2)} du.$$

[6] Let $\varepsilon > 0$ be given. Construct an open subset S of $[0,1]$ with Lebesgue measure less than ε so that the closure of S is $[0,1]$.

[7] Let f be a real valued Lebesgue integrable function defined on the real space. Compute

$$\lim_{n\to\infty} \int_{\mathbb{R}} f(x) \sin(nx) d\lambda$$

(λ being the Lebesgue measure). Justify all your steps!

DEPARTMENT OF MATHEMATICAL SCIENCES
Florida Atlantic University
Analysis Qualifier-Spring 2021

Problem 1

Consider the set $X := \{(x_n)_{x\in\mathbb{N}} | x_n \in [0,1]\}$, equipped with the metric $d : X \times X \to \mathbb{R}$ defined by $d((x_n),(y_n)) = \sup_{n\in\mathbb{N}} |x_n - y_n|$. Let $f : X \to \mathbb{R}$ be uniformly continuous. Show that f is bounded. Does the result hold if f is continuous but not uniformly continuous?

Problem 2

For each $n \in \mathbb{N}$ define $f_n : [0,1] \to \mathbb{R}$ by

$$f_n(x) = \int_0^1 g(x,y)\, ny^n \, dy$$

for each $x \in [0,1]$, where $g(x,y) : \mathbb{R}^2 \to \mathbb{R}$ is continuous. Show that $(f_n)_{n\in\mathbb{N}}$ has a uniformly convergent subsequence.
Hint: Apply Arzelà-Ascoli.

Problem 3

Let $K \subset \mathbb{R}^n$. Show that if every continuous function $f : K \to \mathbb{R}$ is bounded, then K is compact.

Problem 4

Let $A \subset \mathbb{R}$ be a Lebesgue measurable set. Show that if $0 \leq b \leq m(A)$, then there exists a Lebesgue measurable set $B \subset A$ with $m(B) = b$.

Problem 5

If r_n is an enumeration of rational numbers in $\mathbb{R}$ then $\mathbb{R} \setminus \bigcup_{n=1}^{\infty}(r_n - 1/n^2, r_n + 1/n^2)$ is not empty. Prove or find a counterexample.

Problem 6

For each $n \in \mathbb{N}$ let f_n be Lebesgue measurable and assume $\int_{\mathbb{R}} |f_n| \leq 1$. Consider the function defined by

$$f(x) := \begin{cases} \lim_{n\to\infty} f_n(x), & \text{if the limit exists} \\ 0, & \text{otherwise.} \end{cases}$$

Prove that f is Lebesgue measurable and that $\int_{\mathbb{R}} |f| \leq 1$.

Problem 7

(a) Let

$$f_n(x) := \frac{x}{1+x^n}, \quad x \geq 0.$$

Show that the sequence of functions converges pointwise and find the pointwise limit. Is the convergence uniform on $[0, \infty)$?

(b) Compute

$$\lim_{n\to\infty} \int_0^\infty f_n(x)\, dx$$

DEPARTMENT OF MATHEMATICAL SCIENCES
Florida Atlantic University
Analysis Qualifier-Fall 2020

Problem 1

Let $f : [1, \infty) \to \mathbb{R}$ be a continuous function such that $\lim_{x\to\infty} f(x) = \alpha$, i.e. for every $\epsilon > 0$ there exists $M > 0$ such that $|f(x) - \alpha| < \epsilon$ for all $x > M$. Prove that f is uniformly continuous.

Problem 2

Let $(f_n)_{n=1}^{\infty}$ be a sequence of twice differentiable functions on [0,1] such that $f_n(0) = f_n'(0) = 0$ for all n and such that $|f_n''(x)| \leq 1$ for all $x \in [0, 1], n \in \mathbb{N}$. Prove that there is a subsequence $(f_{n_k})_{k=1}^{\infty}$ which converges uniformly on $[0, 1]$.

Problem 3

Assume $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuous. Prove the following two statements are equivalent.

1. $f^{-1}(K)$ is compact for all compact subsets of $\mathbb{R}^n$.
2. $\lim_{|x|\to\infty} |f(x)| = \infty$.

Problem 4

Let $\alpha > 2$ be a real number. Define

$$E = \{x \in [0, 1] \mid |x - p/q| < 1/q^{\alpha} \text{ for infinitely many } p, q \in \mathbb{N}^2\}.$$

Prove that $m(E) = 0$. Hint: Compute the measure of $E_{p,q} = \{x \in [0, 1] \mid |x - p/q| < 1/q^{\alpha}\}$ and apply Borel-Cantelli.

Problem 5

True or False: If the boundary of a set $X \subset \mathbb{R}^d$ has outer measure 0, then X is measurable. Prove or find a counterexample.

Problem 6 Prove that if $f : [0,1] \to \mathbb{R}$ is a continuous function, then

$$\lim_{n\to\infty} \int_0^1 n\, x^n\, f(x) = f(1).$$

Problem 7

Assume $f_n : [0,1] \to [0,\infty)$ is integrable for each n and $(f_n)_{n=1}^{\infty}$ converges pointwise a.e. to f. Prove that

$$\lim_{n\to\infty} \int_{[0,1]} f_n(x) e^{-f_n(x)} dx = \int_{[0,1]} f(x) e^{-f(x)} dx$$

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Analysis Qualifying Exam – Spring 2020

[1] Let $\sigma \colon \mathbb{N} \to \mathbb{N}$ be a one-to-one and onto permutation of the natural numbers. For (E, d) a metric space, prove that if the sequence $\{x_n\}_{n=1}^{\infty}$ converges to x in E, then the permuted sequence $\{x_{\sigma(n)}\}_{n=1}^{\infty}$ also converges to x.

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[2] Let $\{x_n\}_{n=1}^{\infty}$ be a bounded sequence of real numbers. Prove that

$$\liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n.$$

Give an example where the inequality is strict.

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[3] Give an example of a metric space (E, d) and a subset K that is closed and bounded in E but is not a compact subspace of E. Prove that your example satisfies the stated properties.

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[4] Let $K \subset \mathbb{R}$ be compact. Prove that the sequence of functions $\{f_n\}_{n=1}^{\infty}$ defined by $f_n(x) = x/n$ for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$ is uniformly convergent on K but not uniformly convergent on K^c.

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[5] Recall that a metric space, (X, d), is *totally bounded* if for every $\epsilon > 0$ there exists a finite set of points $\{x_1, \ldots, x_n\} \subseteq X$ such that

$$X \subseteq \bigcup_{i=1}^{n} N_\epsilon(x_i).$$

Theorem: If every sequence in X contains a Cauchy subsequence, then X is totally bounded.

(i) State the definition that a sequence, $\{x_n\}_{n=1}^{\infty} \subseteq (X, d)$, is a Cauchy sequence.

(ii) State the contrapositive of the above theorem.

(iii) Prove the above theorem using the contrapositive formulation in (ii).

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[6] Let E_1, E_2 be compact subsets of $\mathbb{R}$ such that $E_1 \subset E_2$.
(a) Prove that $m((E_2 - E_1) \cap [-t, t])$ is a continuous function of $t \geq 0$.
(b) Prove that for every $c \in \mathbb{R}$ with $m(E_1) \leq c \leq m(E_2)$ there exists a compact set E such that $E_1 \subset E \subset E_2$ and $m(E) = c$.

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[7] Prove that the function $F \colon \mathbb{R} \to \mathbb{R}$ defined by

$$F(t) = \int_0^\infty e^{-x} \cos(xt)\, dx \quad \text{for } t \in \mathbb{R}$$

is continuous.

Analysis Qualifying Exam – Fall 2019

[1] Let $\{a_n\}$ and $\{b_n\}$ be bounded sequences of real numbers. Prove that

$$\limsup_{n\to\infty}(a_n + b_n) \le \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n.$$

Give an example where equality does not hold.

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[2] Let (E, d) be a metric space and $f\colon E \to \mathbb{R}$. Suppose $E = X \cup Y$ where X, Y are both open in E and the restrictions $f|_X\colon X \to \mathbb{R}$ and $f|_Y\colon Y \to \mathbb{R}$ are continuous. Prove that f is continuous on E.

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[3] Let (E, d) be a compact metric space and $f\colon E \to \mathbb{R}$ be a continuous function. Prove that f is uniformly continuous.

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[4] Show that the sequence of functions $f_n\colon [0,1] \to \mathbb{R}$

$$f_n(x) = e^{\sin(x+n^2)} + \frac{1}{n}\sin(e^{x+n^2})$$

has a uniformly convergent subsequence.

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[5] Define

$$f_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!}.$$

Compute the limit

$$\lim_{n\to\infty} \int_0^n f_n(x) e^{-2x}\, dx$$

and justify all steps of your solution.

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[6] Let E_k be a sequence of measurable subsets of $\mathbb{R}$ such that

$$\sum_{k=1}^{\infty} m(E_k) < \infty.$$

Show that

$$\{x \in \mathbb{R} \mid x \in E_k \text{ for infinitely many } k\} = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k$$

and that this is a set of measure zero.

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[7] Let $f\colon \mathbb{R} \to \mathbb{R}$ be a Lebesgue integrable function. Show that for every $\epsilon > 0$ there exists $M > 0$ such that

$$\left| \int_{(-\infty,-M)\cup(M,\infty)} f(x)\, dx \right| < \epsilon \quad \text{and} \quad \left| \int_{f^{-1}((-\infty,-M)\cup(M,\infty))} f(x)\, dx \right| < \epsilon$$

Analysis Qualifying Exam
Spring 2019

1. Let (X, d) be a metric space and $A \subset X$. For each $x \in X$ define the distance between x and A by

$$\operatorname{dist}(x, A) = \inf_{a \in A} \{d(x, a)\}.$$

   (a) Show that
$$|\operatorname{dist}(x, A) - \operatorname{dist}(y, A)| \leq d(x, y).$$

   (b) Prove that $\operatorname{dist}(\cdot, A) \colon X \to [0, \infty)$ is a continuous function.

   (c) Suppose A, B are compact, disjoint subsets of X. Prove that there exists $\gamma > 0$ such that

$$\operatorname{dist}(b, A) \geq \gamma \ \text{ for all } b \in B.$$

2. Let $K \subset \mathbb{R}$ be a compact set and let U be an open set that contains K. Prove that there is an $\epsilon > 0$ such that for every $x \in K$, $(x - \epsilon, x + \epsilon) \subset U$.

3. Suppose $f : [0, 1] \to \mathbb{R}$ is continuous. Prove that the graph

$$G := \{(x, y) \in [0, 1] \times \mathbb{R} : y = f(x)\}$$

   is a compact subset of the metric space $[0, 1] \times \mathbb{R}$ with Euclidean topology.

4. Let $x \in \mathbb{R}$ and
$$f_n(x) = \frac{x}{1 + nx^2} \quad n = 1, 2, 3, \ldots$$

   (a) Show that $\{f_n\}$ converges uniformly to a function f.

   (b) Show that the equation

$$f'(x) = \lim_{n \to \infty} f_n'(x)$$

   is correct if $x \neq 0$ but false if $x = 0$.

5. Let f be a measurable function, and let $f = g$ a.e. Then show that g is also measurable.

6. Suppose f is a non-negative integrable function, and set

$$A = \{x | f(x) = +\infty\}.$$

Show that $\mu(A) = 0$, that is the measure of the set A is zero.

7. Let μ denote the Lebesgue measure on $\mathbb{R}$. Suppose f is a bounded measurable function satisfying $\mu(I_n) < n$ for each n, where I_n denotes the set

$$I_n := \{x \in \mathbb{R} : |f(x)| > 1/n^3\}.$$

Prove that $\int |f| < \infty$, i.e., f is integrable.

Hint: You are allowed to assume the truth of the p-series test, i.e., that the numerical series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{n^p}$ converges for any $p > 1$.

Analysis Qualifying Exam   Fall 2018

1. State the following definitions

   (a) A metric space is sequentially compact if

   (b) A metric space is complete if

   (c) A metric space is totally bounded if

   (d) Show that if a metric space is totally bounded and complete, then it is sequentially compact.

2. Let (X, d) be a metric space with disjoint, nonempty, closed subsets $A, B \subset X$. Show that the function $V : X \to [0, 1]$ defined by

$$V(x) = \frac{\mathrm{dist}(x, A)}{\mathrm{dist}(x, A) + \mathrm{dist}(x, B)}$$

   is continuous. Then prove that any connected metric space containing at least two points is uncountable.

3. Let X, Y be metric spaces with Y complete, A be a dense subset of X, and $f : A \to Y$ be a uniformly continuous function. Prove that there exists a uniformly continuous function $g : X \to Y$ such that $g(a) = f(a)$ for all $a \in A$.

4. Let $f : [0, \infty) \to \mathbb{R}$ be bounded and continuous. Suppose $\lim_{h\to\infty} \frac{1}{h} \int_0^h f(x)dx$ exists. Prove that

$$\lim_{h\to\infty} \frac{1}{h} \int_0^h f(x)dx \le \limsup_{x\to\infty} f(x).$$

5. Let $\{f_k(x)\}$ be a sequence of measurable functions defined on a measurable set $E \subset \mathbb{R}$ of finite measure such that $f_k(x) : E \to \mathbb{R}$ for each k. If $|f_k(x)| \le M_x < \infty$ for all k and $x \in E$, show that for every $\epsilon > 0$, there exists a closed $F \subset E$ and a finite M such that $m(E \setminus F) < \epsilon$ and $|f_k(x)| \le M$ for all k and $x \in F$.

6. Let $f \in L^1(\mathbb{R})$ and $h \in \mathbb{R}$. Show (without using the change of variables rule from Calculus, which is inappropriate for this problem) that

$$\int_{\mathbb{R}} f(x)dx = \int_{\mathbb{R}} f(x-h)dx$$

7. Let $g \in L^1(0,\infty)$, and consider the function

$$f(x) = \int_0^\infty e^{-xy} g(y)\, dy.$$

for $x \in (0,\infty)$. Prove that f is differentiable for all $x > 0$ and compute $f'(x)$.

Analysis Qualifying Exam Spring 2018

1. Suppose X is a compact metric space. Given an open cover $\mathcal{U}$ of X, show that there exists a $\delta > 0$ such that for every $x \in X$ the set

$$\{y \in X : d(x, y) < \delta\}$$

is contained in some member of $\mathcal{U}$.

2. Suppose that X is a compact metric space and $f : X \to X$ is an isometry. Show that $f(X) = X$. i.e., f is onto.

Hint: Suppose that f is not onto. Starting with a point $y \in X$ not in the image, iterate f and consider the sequence of iterates $f^n(y)$.

3. Prove that the sequence of functions

$$f_n(x) = \cos(x + n) + \frac{\cos(1 + e^{nx})}{n}$$

has a subsequence that converges uniformly on $[0, 1]$.

4. Suppose that $A \subset \mathbb{R}$ is Lebesgue measurable with Lebesgue measure $m(A) = 1$. Show that there is a set $B \subset A$ such that $m(B) = 1/2$.

5. (a) State Fatou's Lemma, the Monotone Convergence Theorem, and Lebesgue's Dominated Convergence theorem.

(b) Provide an example where the inequality in Fatou's Lemma is strict.

6. Suppose f is a continuous function on $\mathbb{R}$ satisfying $\lim_{x\to+\infty} f(x) = L \in \mathbb{R}$. Prove that the following limit exists and compute its value:

$$\lim_{n\to\infty} \int_0^1 f(nx)dx.$$

7. Suppose f is Lebesgue integrable on $\mathbb{R}$. Prove that

$$\lim_{n\to\infty} \int f(x)\cos(nx)\, dx = 0.$$

Analysis Qualifying Exam, Fall 2017

1. Prove that the intersection

$$\bigcap_{j=1}^{\infty} K_j$$

of a nested sequence

$$K_1 \supset K_2 \supset K_3 \supset \ldots$$

of non-empty compact sets (contained in a metric space) is non-empty and compact.

2. Prove that the function

$$f(x) = \frac{1}{x^2}$$

is uniformly continuous on the interval $[3, \infty)$ and is not uniformly continuous on the interval $(0, 3)$.

3. Let $C^0[a, b]$ denote the metric space of continuous function on the interval $[a, b]$ with the metric $d(f, g) := \sup\{|f(x) - g(x)| : x \in [a, b]\}$.

(a) Use the Stone-Weierstrass theorem to prove that the set of even polynomials, i.e., functions of the form $p(x) = a_0 + a_2x^2 + \ldots + a_{2n}x^{2n}$ is dense in $C^0[0, 1]$.

(b) Prove that the set of even polynomials is not dense in the space $C^0[-1, 1]$.

4. Suppose $E_k \subset \mathbb{R}$ is measurable for each $k = 1, 2, \ldots,$ and

$$m(E_k) < 1/2^k.$$

Prove that for every $\varepsilon > 0$ there exists an N such that

$$m\left(\bigcup_{k=N}^{\infty} E_k\right) < \varepsilon.$$

5. Suppose f is a non-negative, Lebesgue integrable function on $\mathbb{R}$. Prove that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\int_E f(x)dx < \varepsilon,$$

for every measurable set $E \subset \mathbb{R}$ that satisfies $m(E) < \delta$.

6. Prove that the limit exists and compute its value:

$$\lim_{n\to\infty} \int_{1/2}^{\infty} \frac{1}{x^2} \frac{x^n}{x^2 + x^n} dx.$$

7. Suppose $g \in L^1([1, \infty))$, and for $x > 1$ consider the function

$$f(x) = \int_1^{\infty} e^{-\frac{x}{y}} g(y)\, dy.$$

Fix $x > 1$ and justify the following formula (starting from the definition of the derivative as the limit of a difference quotient):

$$f'(x) = \int_1^{\infty} -\frac{1}{y} e^{-\frac{x}{y}} g(y)\, dy.$$

Analysis Qualifying Exam

Spring 2017

Problem 1: Let f be differentiable on $\mathbb{R}$. Suppose that there exists $M > 0$ such that $|f(k)| \leq M$ for each integer k, and $|f'(x)| \leq M$ for all $x \in \mathbb{R}$. Show that f is bounded, i.e., there exists $B > 0$ such that $|f(x)| \leq B$ for all $x \in \mathbb{R}$.

Problem 2: For each of the following descriptions, give an example of such a sequence of real numbers or explain that it is not possible.

a. An unbounded sequence that has a bounded subsequence that does not converge and also has a subsequence that does converge.

b. A sequence that has a monotone subsequence and a bounded subsequence but does not have a convergent subsequence.

c. A sequence that has subsequences converging to two different values not appearing in the sequence.

Problem 3: Suppose $f : [0, 1] \to \mathbb{R}$ is continuous. Prove that the graph $G := \{(x, y) \in [0, 1] \times \mathbb{R} : y = f(x)\}$ is compact. Hint: Use the sequential criterion for compactness.

Problem 4: Prove that the sequence $h_n(x) = \frac{x}{1+x^n}$ converges uniformly on the interval $[2, \infty)$ but does not converge uniformly on $[0, \infty)$.

Problem 5: Compute

$$\lim_{n\to\infty} \int_0^\infty e^{-nx} \sin(x/n) dx.$$

Justify the steps.

Problem 6: Suppose that $f \in L^1(\mathbb{R})$.

- For $\tau \in \mathbb{R}$, show that

$$\int_{\mathbb{R}} f(x)\, dx = \int_{\mathbb{R}} f(x - \tau)\, dx.$$

- Show that

$$\lim_{h\to 0} \|f_h - f\|_{L^1(\mathbb{R})} = 0,$$

where f_h is defined by $f_h(x) = f(x + h)$.

Problem 7: Recall that if $f, g \in L^1(\mathbb{R})$ then the convolution of f and g defined by

$$(f * g)(x) := \int_{\mathbb{R}} f(y)g(x-y)\,dy,$$

exists for almost every $x \in \mathbb{R}$ and $f * g \in L^1(\mathbb{R})$. Moreover, convolution defines a commutative binary operation on L^1 with $\|f * g\|_{L^1(\mathbb{R})} \leq \|f\|_{L^1(\mathbb{R})}\|g\|_{L^1(\mathbb{R})}$, i.e. $L^1(\mathbb{R})$ is a commutative Banach algebra under the operation of convolution.

(A) Suppose that $f \in L^1(\mathbb{R})$ and that $g\colon \mathbb{R} \to \mathbb{R}$ is a bounded function. Prove that $(f * g)(x)$ is uniformly continuous. Hint: you can use the results of problem 5 even if you did not complete that problem.

(B) Prove that there is no function $\delta \in L^1(\mathbb{R})$ having that

$$(\delta * f)(x) = f(x),$$

for all $f \in L^1(\mathbb{R})$. Hint: let f be the indicator function of an interval (or just take f to be your favorite discontinuous, bounded, L^1 function). Consider both sides of $\delta * f = f$, taking into account your choice and the results of part (A).

Problem 8:

- Give an example of a sequence of measurable functions $f_n\colon [0,1] \to \mathbb{R}$ which converge to zero in $L^1(\mathbb{R})$ norm, but which do not converge pointwise for any $x \in [0,1]$, i.e. a sequence of functions which has that

$$\lim_{n\to\infty} \|f_n\|_{L^1(\mathbb{R})} = 0,$$

but that the sequence $\{f_n(x)\}_{n=0}^{\infty}$ diverges for all $x \in [0,1]$. (This shows that L^1 convergence does not imply pointwise convergence at even a single point).

- Let $f_n\colon \mathbb{R} \to \mathbb{R}$ be a sequence of measurable functions with $f_n \in L^1(\mathbb{R})$ for each n. Suppose that

$$\|f_n - f_{n-1}\|_{L^1(\mathbb{R})} \leq 2^{-n}.$$

Prove that there is a measurable function $f\colon \mathbb{R} \to \mathbb{R}$ having that $f \in L^1(\mathbb{R})$, and that f_n converges to f in L^1 norm, i.e.

$$\lim_{n\to\infty} \|f_n - f\|_{L(\mathbb{R})} = 0.$$

Justify your steps.

- Let f_n and f be as in the previous problem. Show that you actually have

$$\lim_{n\to\infty} f_n(x) = f(x),$$

a.e., in other words the sequence is pointwise convergent almost everywhere. (This shows that L^1 convergence, plus rates of convergence, does imply pointwise convergence).

Analysis Qualifying Exam

Fall, 2016

Problem 1:

- Let $a, b \in \mathbb{R}$ and suppose that $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ are sequences of real numbers with

$$\lim_{n\to\infty} a_n = a, \qquad \text{and} \qquad \lim_{n\to\infty} b_n = b.$$

  Prove that

$$\lim_{n\to\infty} a_n b_n = ab.$$

- Prove that a bounded increasing sequence of real numbers converges to a limit.

Problem 2: Suppose that X, Y are metric spaces and that X is compact. If $f\colon X \to Y$ is continuous prove that $f(X)$ is compact.

Problem 3: Suppose that f is a positive, continuous function on $\mathbb{R}$ such that

$$\lim_{|x|\to\infty} f(x) = 0.$$

Prove that f is uniformly continuous.

Problem 4:

- Give an example of a function which is Lebesgue integrable but not Riemann integrable. Explain your reasoning.
- Give an example of a sequence of bounded, continuous functions $f_n \colon \mathbb{R} \to \mathbb{R}$ having that
$$\lim_{n\to\infty} f_n(0) = \infty,$$
but that
$$\lim_{n\to\infty} \|f_n\|_{L^1(\mathbb{R})} = 0.$$
(So: divergence at a point does not imply divergence in L^1).
- Given an example of a sequence of functions $f_n \colon \mathbb{R} \to \mathbb{R}$ having that
$$\|f_n\|_{L^1(\mathbb{R})} = 1$$
for every $n \in \mathbb{N}$ but which converges pointwise to zero. (So: pointwise convergence to zero does not imply L^1 convergence to zero).

Problem 5: Suppose that $f \in L^1(\mathbb{R})$.

- For $\tau \in \mathbb{R}$, show that
$$\int_{\mathbb{R}} f(x)\,dx = \int_{\mathbb{R}} f(x-\tau)\,dx.$$
- Show that
$$\lim_{h\to 0} \|f_h - f\|_{L^1(\mathbb{R})} = 0,$$
where f_h is defined by $f_h(x) = f(x+h)$.

Problem 6: Recall that if $f, g \in L^1(\mathbb{R})$ then the convolution of f and g defined by

$$(f * g)(x) := \int_{\mathbb{R}} f(y)g(x-y)\,dy,$$

exists for almost every $x \in \mathbb{R}$, i.e. $f * g \in L^1(\mathbb{R})$. Moreover, convolution defines a commutative binary operation on L^1 with $\|f * g\|_{L^1(\mathbb{R})} \leq \|f\|_{L^1(\mathbb{R})}\|g\|_{L^1(\mathbb{R})}$, i.e. $L^1(\mathbb{R})$ is a commutative Banach algebra under the operation of convolution.

Question: Define the functions $\phi_n \colon \mathbb{R} \to \mathbb{R}$ by

$$\phi_n(x) = \begin{cases} n/2 & \text{if } |x| \leq 1/n \\ 0 & \text{otherwise} \end{cases}.$$

Prove that for all $f \in L^1(\mathbb{R})$,

$$\lim_{n\to\infty} \|f * \phi_n - f\|_{L^1} = 0.$$

Hint: You are allowed to use the results stated in Problem 5 (even if you did not do that problem).

Problem 7: For $f \in L^1(\mathbb{R})$ and $x \in \mathbb{R}$, show (while justifying each step) that the derivative of the function

$$F(x) = \int_{\mathbb{R}} \sin(y) f(x-y)\,dy,$$

is given by the formula

$$F'(x) = \int_{\mathbb{R}} \cos(y) f(x-y)\,dy.$$

(Hint: think of using the dominated convergence theorem).

Analysis Qualifying Exam
January 21, 2016

INSTRUCTIONS

- Number all your pages and **write only on one side of the paper.** Anything written on the second side of a page will be ignored.
- Write your name at the top of each page.

1. Let $f : \mathbb{R} \to \mathbb{R}$ satisfy $f(x+y) = f(x)+f(y)$ for all $x, y \in \mathbb{R}$ and $f(1) = 1$. Prove:
   (a) $f(x) = x$ for all rational numbers x.
   (b) Assume, in addition, that f is continuous at 0. Prove that then $f(x) = x$ for all $x \in \mathbb{R}$.
2. We say that a family of subsets of a metric space (X, d) is *locally finite* if for each $p \in X$ there is an open set V such that $p \in V$ and V only intersects a finite number of the sets F_n. Prove: If $\{F_n\}$ is a locally finite family of **closed** sets, then $\bigcup_{n=1}^{\infty} F_n$ is closed.
3. Let X be the metric space consisting of all sequences $a = (a_1, a_2, \ldots)$ of real numbers such that $\sum_{n=1}^{\infty} |a_n| < \infty$, with the distance function defined by
$$d(a, b) = \sum_{n=1}^{\infty} |a_n - b_n|$$
   if $a = (a_1, a_2, \ldots)$, $b = (b_1, b_2, \ldots)$.
   (a) Prove $\bar{B}(0, 1) = \{a = (a_1, a_2, \ldots) : d(a, 0) \leq 1\}$ is **not** compact.
   (b) Let $C = \{a = (a_1, a_2, \ldots) : |a_n| \leq 1/n^2 \text{ for } n \in \mathbb{N}\}$. Prove C is compact.
4. Let X be a metric space and let $f_n : X \to \mathbb{R}$ for each $n \in \mathbb{N}$. We say that the sequence $\{f_n\}$ is *locally uniformly convergent* if for every $p \in X$ there exists an open set U in X such that $p \in U$ and the sequence of restrictions $\{f_{nU}\}$ converges uniformly on U. Prove: If X is compact and the sequence $\{f_n\}$ converges locally uniformly, then it is uniformly convergent.
5. Let $f : [-1, 1] \to \mathbb{R}$ be continuous and even ($f(-x) = f(x)$ for all $x \in [-1, 1]$). Prove: For each $\epsilon > 0$ there exists a polynomial p such that $|f(x) - p(x^2)| < \epsilon$ for all $x \in [-1, 1]$.
6. Prove or disprove: There exists a **closed** subset F of $\mathbb{R}$ such that F has positive measure and $F \cap \mathbb{Q} = \emptyset$.
7. Evaluate, justifying all steps:
$$\lim_{n\to\infty} \int_0^{\infty} \frac{n \sin \frac{x}{n}}{x(1 + x^2)}\, dx.$$
   Hint: You may use that $|\sin x/x| \leq 1$ for all $x \in (0, \infty)$.

Analysis Qualifying Exam

August 28, 2015

INSTRUCTIONS

- Read the instructions.
- Number all your pages and **write only on one side of the paper.** Anything written on the second side of a page will be ignored.
- Write your name at the top of each page.
- Clearly indicate which problem you are solving, keep solutions to different problems separate.

1. Let $u_n \geq 0$ for all $n \in \mathbb{N}$. **Prove:** If $\sum_{n=1}^{\infty} u_n$ converges, then $\sum_{n=1}^{\infty} \frac{\sqrt{u_n}}{n}$ also converges.

2. **Prove** there exists a unique differentiable function $\Phi : \mathbb{R} \to \mathbb{R}$ such that $\Phi'(x) = e^{-x^2}$ for all $x \in \mathbb{R}$ and $\Phi(0) = 0$.

3. Let X be a metric space, let $C \subset X$ have the property that if $x, y \in C$, there exists a connected subset A of C such that $x, y \in A$. **Prove:** C is connected.

4. Let $\mathcal{E}$ be an equicontinuous and bounded set of functions from $[0, 1]$ to $\mathbb{R}$. **Prove:** If $\{f_n\}$ is a sequence in $\mathcal{E}$ that converges for each **rational** $x \in [0, 1]$, then $\{f_n\}$ converges uniformly.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be Lebesgue integrable. **Prove** that
$$\lim_{n\to\infty} \int_{\mathbb{R}} f(x) \cos nx \, dx = 0.$$

6. Assume $f : \mathbb{R} \to \mathbb{R}$ is continuous. **Prove:** The inverse image under f of a Borel set is a Borel set.

7. **Prove** that the following limit exists:
$$\lim_{n\to\infty} \int_0^{\infty} \frac{\cos x}{nx^2 + 1/n} dx.$$
Be sure to justify all steps.

   Hint: Change variables by $t = nx$.

Analysis Qualifying Examination – Spring 2015

Your Name: ________________ **Your Z-Number:** ________________

In some cases partial credit may be given, but you should endeavor to fully complete as many problems as possible.

1. Let $\{a_n\}$ be a sequence of real numbers that converges to a. Show that

$$\lim_{n\to\infty} \frac{a_1 + a_2 + \cdots + a_n}{n} = a.$$

2. Consider the series

$$\sum_{n=1}^{\infty} \frac{x}{(1+x)^n}.$$

Show that the series converges for all $x \geq 0$ and that it converges uniformly on the interval $[r, \infty)$ for every $r > 0$. Does the series converge uniformly on $(0, \infty)$? Justify your answer.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function that satisfies, for each $x \in \mathbb{R}$,

$$f(x) \leq \frac{f(x-h) + f(x+h)}{2} \quad \text{for all } h > 0.$$

Show that the maximum value of f on any bounded closed interval $[a, b]$ is attained at one of the endpoints, that is, either $f(a)$ or $f(b)$ is the maximum value of f on the interval $[a, b]$.

4. (a) Show that the inequalities

$$\frac{x}{x+1} < \ln(1+x) < x$$

hold for all $x > 0$.

(b) Define

$$f(x) = \left(1 + \frac{1}{x}\right)^{x} \quad \text{and} \quad g(x) = \left(1 + \frac{1}{x}\right)^{x+1} \quad \text{for all } x > 0,$$

where we define $x^y = e^{y \ln x}$ for all $x > 0$ and $y > 0$ and e is Euler's number (also known as Napier's constant). Show that f is strictly increasing while g is strictly decreasing on the interval $(0, \infty)$, and that $f(x) < e < g(x)$ for all $x > 0$.

5. Let $\{f_n\}$ be a sequence of real-valued functions defined on a compact matric space (X, d) such that $f_n(x_n) \to f(x)$ in $\mathbb{R}$ whenever $x_n \to x$ in X. Assume that f is continuous. Show that $\{f_n\}$ converges uniformly to f.

6. (a) Show that the sequence of functions

$$f_n(x) = e^{-n(nx-1)^2}$$

point-wise converges to zero but not uniformly on $[0, 1]$.

(b) Nevertheless show that

$$\lim_{n\to\infty} \int_0^1 f_n(x)\, dx = 0.$$

7. Let (X, d) be a compact metric space. Assume that $f : X \to X$ is an expansion map, that is, $d(f(x), f(y)) \geq d(x, y)$ for all $x, y \in X$. For every $x \in X$, define $f^2(x) = f(f(x))$, $f^3(x) = f(f^2(x))$, and in general $f^n(x) = f(f^{n-1}(x))$ for $n \geq 2$. Prove the following statements:

(a) For every $x \in X$, we have $d(x, f^{m-n}(x)) \leq d(f^n(x), f^m(x))$ for all positive integers m and n with $m > n$, and that the sequence $\{f^n(x)\}$ contains a subsequence $\{f^{n_k}(x)\}$ such that $f^{n_k}(x) \to x$ as $k \to \infty$.

(b) For every pair of points (x, y), the sequence $\{f^n\}$ contains a subsequence $\{f^{n_k}\}$ such that $f^{n_k}(x) \to x$ and $f^{n_k}(y) \to y$ as $k \to \infty$. (Hint: consider the compact metric space $X \times X$ and the product metric $D((x, y), (u, v)) = d(x, u) + d(y, v)$ and the map $F : X \times X \to X \times X$ defined by $F(x, y) = (f(x), f(y))$)

(c) For all $x, y \in X$, $d(f(x), f(y)) = d(x, y)$, that is, f is an isometry.

Analysis Qualifying Examination Fall 2014

In some cases partial credit may be given, but you should endeavor to fully complete as many problems as possible.

1. Let f be a real-valued differentiable function defined on $(-\infty, +\infty)$. Suppose that f has a bounded derivative. Show that there exist nonnegative constants A and B such that $|f(x)| \le A|x| + B$ for all $x \in (-\infty, +\infty)$.

2. Consider the series
$$\sum_{n=1}^{\infty} \frac{n^2x^2}{1+n^4x^4}.$$
(a) Show that for every $\delta > 0$ the series converges uniformly on the set $\{x : |x| \ge \delta\}$.
(b) Does the series converge uniformly on $(-\infty, \infty)$? Justify your answer.

3. Let f be a continuous real-valued function on $[a, b]$. Suppose that there exists a constant $M \ge 0$ such that
$$|f(x)| \le M \int_a^x |f(t)|\, dt$$
for all $x \in [a, b]$. Show that $f(x) = 0$ for all $x \in [a, b]$.

4. Let A and B be two nonempty subsets of $\mathbb{R}^n$, where $\mathbb{R}^n$ is the n-dimensional Euclidean space equipped with the usual metric. Define $A + B = \{a + b : a \in A,\, b \in B\}$, where $a + b$ is the sum of vectors a and b in the usual sense.
(a) If A and B are compact, show that $A + B$ is compact.
(b) If A and B are connected, show that $A + B$ is connected.
(b) If one of A and B is open, show that $A + B$ is open.

5. Show that the following limit exists and find the limit.
$$\lim_{n\to\infty} \int_0^{\infty} \frac{\cos(x^n)}{1+x^n}\, dx.$$

6. Let f be a real-valued measurable function defined on a bounded measurable set E. Suppose that there exists a $\delta > 0$ such that $\int_F |f| < 1$ whenever F is a measurable subset of E and $m(F) < \delta$. Show that f is Lebesgue integrable on E. Here m denotes the Lebesgue measure.

7. Let E be a measurable subset of $\mathbb{R}$. Define $E^2 = \{x^2 : x \in E\}$. If E has measure zero, show that E^2 also has measure zero.

Analysis Qualifying Examination Spring 2014

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[1] Let $(\mathbb{M}, \rho)$ be a metric space. Suppose that $f : \mathbb{M} \to \mathbb{R}$ is uniformly continuous. Show that if $\{x_n\}$ is a Cauchy sequence in $\mathbb{M}$, then the sequence $\{f(x_n)\}$ is Cauchy in $\mathbb{R}$.

Give an example which shows that the result is not necessarily true if f is assumed to be continuous but not uniformly continuous.

---

[2] Show that the set $A = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ is a compact, connected subspace of the Euclidean space $\mathbb{R}^2$.

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[3] A subset A of a metric space $(\mathbb{M}, \rho)$ is *precompact* if its closure $\mathrm{cl}(A)$ is compact.

Show that if A is precompact, then for every $\epsilon > 0$ there exists a finite covering of A by open balls of radius ϵ **with centers in** A**.**

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[4] Let $\epsilon > 0$. Construct an open subset S of $[0, 1]$ with Lebesgue measure less than ϵ so that the closure of S is $[0, 1]$.

---

[5] Let λ be the Lebesgue measure on $\mathbb{R}$. Suppose E is a Lebesgue measurable subset of $[0, 1]$ with $\lambda(E) = 1$. Show that E is dense in $[0, 1]$.

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[6] Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of Lebesgue measurable functions on a Lebesgue measurable subset E of $\mathbb{R}$ which converges pointwise to a function f. Suppose $\int_E |f_n| \le 1$ for all $n \in \mathbb{N}$. Prove $\int_E |f| \le 1$.

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[7] Compute

$$\lim_{n\to\infty} n \int_0^1 \frac{\sin(u)}{u} e^{-nu} \sin(nu)\, du.$$

Analysis Qualifying Exam.

Sept 6, 2013

Student Name (Print)______________________________

There are 7 questions.

Student Name (Print)______________________________

1. Assume that $\{x_k\}$ is a Cauchy sequence in a metric space, $(\mathbb{M}, \rho)$, and that a subsequence, $\{x_{k_n}\}$, converges. Prove that $\{x_k\}$ converges.

Student Name (Print)______________________________

2. Prove that if $0 < s \leq 1$ then $f(x) = x^s$ is uniformly continuous on $[0, \infty]$.

Student Name (Print)____________________________

3. Assume that $(\mathbb{M}, \rho)$ is a compact metric space, and that $\{\mathcal{G}_\alpha\}$ is an open cover of $\mathbb{M}$. Prove that there exists $\delta > 0$ so that any ball with radius smaller than δ is a subset of at least one of the $\mathcal{G}_\alpha$.

Student Name (Print)___________________________

4. Assume that f and g are continuous non-negative functions on a compact metric space, $(\mathbb{M}, \rho)$ and that $\{x : g(x) = 0\} \subseteq \{x : f(x) = 0\}$. Prove that for any $\varepsilon > 0$ there exists $K(\varepsilon)$ so that for all $x \in \mathbb{M}$,

$$f(x) \leq \varepsilon + K(\varepsilon) g(x).$$

Student Name (Print)____________________________

5. Assume that $(\mathbb{M}, \rho)$ is a complete metric space and that $f : \mathbb{M} \mapsto \mathbb{M}$ is a uniform contraction, that is to say, that there exists $0 < q < 1$ so that $\forall x, y \in \mathbb{M},$.

$$\rho(f(x), f(y)) \leq q\rho(x, y)$$

Prove that there exists a unique $x \in \mathbb{M}$ so that $f(x) = x$.

Student Name (Print)___________________________

6. Assume that f is a real-valued Lebesgue measurable function on $\mathbb{R}$ and for all $\alpha > 0$, $\lambda(\{x: f(x) > \alpha\}) = \lambda(\{x: f(x) < -\alpha\})$ then

$$\lambda(\{x: |f| > \alpha\}) \leq 2e^{-\alpha^2} \int_{-\infty}^{\infty} e^{\alpha f(x)} dx.$$

Note: λ denotes Lebesgue measure on $\mathbb{R}$.

Student Name (Print)____________________________

7. Prove that if f is a Lebesgue measurable function and

$$\int_{-\infty}^{\infty} |f(x)|dx < \infty$$

then $\forall \varepsilon > 0$ there exists a Lebesgue measurable set, $E(\varepsilon)$, so that $\lambda(E(\varepsilon)) < \infty$, $|f|$ is bounded on $E(\varepsilon)$, and

$$\int_{(-\infty,\infty)\setminus E(\varepsilon)} |f(x)|dx < \varepsilon.$$

Analysis Qualifying Examination Spring 2013

Name: ________________

1. Give a proof of Dini's Theorem: let $F_n : [a,b] \to \mathbb{R}$ be an increasing sequence of continuous functions (i.e., $F_{n+1} \geq F_n$ for all n) converging pointwise to a continuous function F. Then (F_n) converges to F uniformly

2. Let $f : \mathbb{R} \to \mathbb{R}^+$ be a continuous integrable function. Show that
$$f \text{ is uniformly continuous} \Leftrightarrow \lim_{|x|\to\infty} f(x) = 0$$

3. Let $f : (0,1] \to \mathbb{R}$ be differentiable such that f' is bounded on $(0,1]$. Prove that $\lim_{x\to 0^+} f(x)$ exists.

4. Show that the series
$$\sum_{n\geq 1} \frac{x^n}{n+1}(1-x)$$
converges uniformly on $[-1,1]$.

5. Let $g : \mathbb{R} \to \mathbb{R}$ be a measurable function. Assume that for each measurable set $B \subset \mathbb{R}$
$$\int_B g d\lambda = 0.$$
Show that $g \equiv 0$ a.e.

6. Compute
$$\lim_{n\to\infty} \int_0^1 \frac{n^4 u^2 e^{-nu} du}{1+n^2 u}$$

Analysis Qualifying Examination Fall 2012

Solve as many problems as you can. You do not have to solve all to pass the qualifying exam.

1. Suppose that each f_n is increasing and continuous on $[a, b]$, and that the series

$$F(x) = \sum_{n=1}^{\infty} f_n(x)$$

converges for every $x \in [a, b]$. Prove that F is continuous on $[a, b]$.

2. Let $g : \mathbb{R} \to \mathbb{R}$ be a C^2 function. Assume that g and g'' are both bounded on $\mathbb{R}$. Show that g' is bounded on $\mathbb{R}$

Hint: Show that there exists a sequence (b_n), $n \in \mathbb{Z}$ such that $n < b_n \leq n + 1$ and $(g'(b_n))$ is a bounded sequence.

3. Define

$$u_n = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdots 2n}$$

(a) Determine the radius of convergence R of the series $\sum_{n\geq 1} u_n x^n$

(b) Study the convergence at $x = -R$ (hint: study $-\log(u_n)$).

(c) Study the convergence at $x = R$.

4. Let (a_n) be a non negative sequence. Show that

$$\sum_{n\geq 1} a_n \text{ converges} \Rightarrow \sum_{n\geq 1} \frac{\sqrt{a_n}}{n} \text{ converges.}$$

Is the converse true?

5. Compute

$$\lim_{n\to\infty} \int_0^n \left(1 + \frac{x}{2n}\right)^n e^{-x} dx.$$

6. Let E be a measurable set of finite measure. For each $x \in \mathbb{R}$, let $f(x) = m(E \cap (-\infty, x])$. Prove that f is uniformly continuous on $\mathbb{R}$, and that $f(n) \to m(E)$ as $n \to \infty$.

Analysis Qualifying Examination Spring 2012

Name: ________________ **Z-Number:** ________________

In some cases partial credit may be given, but you should endeavor to fully complete as many problems as possible.

1. Consider the series
$$f(x) = \sum_{n=1}^{\infty} \frac{nx}{1+n^3x^2}.$$
(a) Show that the series converges for every real number x.
(b) Show that for any $\delta > 0$ the series converges uniformly for $|x| \geq \delta$.
(c) Show that the function f defined above for all real numbers x is continuous at every non-zero x. Is it continuous at $x = 0$? Justify your answer.

2. Let f be a real-valued differentiable function defined on $(-r, r)$, where r is a positive number. Show that f is an even function if and only if its derivative f' is an odd function.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the equation $f(x+y) = f(x)f(y)$ for all real numbers x and y.
(a) Show that $f(0) = 0$ or $f(0) = 1$.
(b) Show that if $f(0) = 1$, then
$$f\left(\frac{m}{n}\right) = (f(1))^{\frac{m}{n}}$$
for all integers m and n, where n is non-zero.
(c) Show that f is either the zero function or there exists a positive number a such that $f(x) = a^x$ for all real numbers x.

4. Let $f : [0,1] \to \mathbb{R}$ be differentiable function satisfying the conditions: $f(0) = 0$, $|f'(x)| \leq |f(x)|$ for all $0 < x < 1$. Prove that f is the zero function.

5. Let f be a real-valued continuous function defined on $[0,1]$. Show that
$$\lim_{n\to\infty} \int_0^1 f(x^n)\, dx = f(0).$$

6. Let A be a Lebesgue measurable subset of $\mathbb{R}$ with $m(A) > 0$, where m denotes the Lebesgue measure. Show that for any $0 < \delta < m(A)$ there exists a measurable subset B of A such that $m(B) = \delta$. Hint: consider the function $f(x) = m(A \cap [-x, x])$ for all $x > 0$.

Analysis Qualifying Examination **Fall 2011**

Your Name: ________________ **Your Z-Number:** ________________

In some cases partial credit may be given, but you should endeavor to fully complete as many problems as possible.

1. Prove that the function $f(x) = \sqrt{x}$ is uniformly continuous on $[0, \infty)$.

2. Let $a < b$. Suppose that the function $f : [a, b] \to \mathbb{R}$ is bounded and Riemann integrable on $[c, b]$ for every $a < c < b$. Prove that f is Riemann integrable on $[a, b]$ and $\int_a^b f(x)\,dx = \lim_{c \to a} \int_c^b f(x)\,dx$.

3. (a) Is there a closed uncountable subset of $\mathbb{R}$ which contains no rational numbers? Prove your assertion. (b) Is there an infinite compact subset of $\mathbb{Q}$? Prove your assertion. Here $\mathbb{R}$ denotes the set of all real numbers equipped with the usual metric and $\mathbb{Q}$ is the set of all rational numbers.

4. Consider the power series
$$\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} x^n.$$
(a) Prove that the series converges for $|x| < 1$ and diverges for $|x| > 1$. (b) Investigate the convergence and divergence of the series for $x = \pm 1$.

5. Let (X, d) be a metric space and let A and B be two nonempty subsets of X such that $A \cap B = \emptyset$. Prove that if A is closed and B is compact, then $d(A, B) > 0$, where $d(A, B)$ denotes the distance between A and B.

6. Let $a_1 = 0$ and for every positive integer $n \geq 2$, let
$$a_n = \int_0^{\infty} \frac{x^{\frac{1}{n}}}{1 + x^2}\,dx.$$
Show that the sequence $\{a_n\}$ converges and find its limit.

ANALYSIS QUALIFYING EXAMINATION

January 10, 2011

Solutions to the problems are posted at http://math.fau.edu/AnQualifiers/anqua_Jan2011sol.pdf

1. Let A, B be non-empty sets of real numbers such that $a \le b$ for all $a \in A$, $b \in B$. Prove the following two statements are equivalent:

   (a) $\sup A = \inf B$.

   (b) For every $\epsilon > 0$ there exist $a \in A$ and $b \in B$ such that $b - a < \epsilon$.

2. Let $f : [0, \infty) \to \mathbb{R}$ be uniformly continuous and assume that

$$\lim_{b\to\infty} \int_0^b f(x)\, dx$$

   exists and is finite. Prove that $\lim_{x\to\infty} f(x) = 0$.

3. Let $\psi : \mathbb{R} \to \mathbb{R}$ be defined by

$$\psi(x) = \begin{cases} 0 & \text{if } x < 0, \\ x & \text{if } 0 \le x < 1, \\ 1 & \text{if } x \ge 1. \end{cases}$$

   Consider the series $\displaystyle\sum_{k=1}^{\infty} \frac{\psi(kx)}{k(1+kx^2)}$.

   (a) Prove the series converges for all $x \in \mathbb{R}$.

   (b) Let $f(x) = \displaystyle\sum_{k=1}^{\infty} \frac{\psi(kx)}{k(1+kx^2)}$. Prove $\limsup_{x\to 0+} f(x) > 0$.

   Hint: $f(x) \ge \displaystyle\sum_{k=1}^{n} \frac{\psi(kx)}{k(1+kx^2)}$; estimate for $x = 1/n$.

   (c) Prove f is continuous on $(-\infty, 0) \cup (0, \infty)$ but discontinuous at 0.

4. Let $\mathcal{A}$ consist of all functions from $[0, \pi]$ to $\mathbb{R}$ that are finite linear combinations of elements of the set $\{\sin(nx) : n \in \mathbb{N}\}$; that is, $f \in \mathcal{A}$ if and only if $f(x) = \sum_{k=1}^{n} a_k \sin(kx)$ for some $n \in \mathbb{N}$, $a_1, \ldots, a_n \in \mathbb{R}$.

   (a) Prove: If $f : [0, \pi] \to \mathbb{R}$ is continuous and satisfies $f(0) = f(\pi) = 0$, then f can be approximated uniformly by a sequence in $\mathcal{A}$.

   Hint: Add the constant function 1 to $\mathcal{A}$; take it away again later on.

   (b) Prove: If $f : [0, \pi] \to \mathbb{R}$ is continuous and satisfies $\displaystyle\int_0^{\pi} f(x) \sin nx\, dx = 0$ for all $n \in \mathbb{N}$, then $f(x) = 0$ for all $x \in [0, \pi]$.

5. Let $f_n : \mathbb{R} \to \mathbb{R}$ be measurable for $n = 1, 2, \ldots$. Let $a_n = \displaystyle\int_{\mathbb{R}} |f_n|$ for $n = 1, 2, \ldots$ and assume that $\sum_{n=1}^{\infty} a_n < \infty$. Prove: The series $\sum_{n=1}^{\infty} f_n$ converges almost everywhere.

6. Compute, and justify your computation,

$$\lim_{n\to\infty} \int_0^1 \frac{\cos(x^n)}{1 + x^n}\, dx.$$

Analysis Qualifier

August 19, 2010

INSTRUCTIONS:

- Write only on one side of each of the sheets you hand in. Anything written on the back of a sheet might be ignored.
- Write your name on each sheet.
- Write clearly. A completely solved problem is worth more than several problems left half done.

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1. Let $\{a_n\}$ be a sequence of real numbers and let S be the set of all limits of subsequences of $\{a_n\}$; that is, $x \in S$ if and only if $x \in \mathbb{R}$ and there exists a sequence of positive integers $\{n_k\}$ such that $n_1 < n_2 < n_3 < \cdots$ and such that $\lim_{k\to\infty} a_{n_k} = x$. Prove: S is a closed subset of $\mathbb{R}$.

2. Let $\sum_{n=1}^{\infty} a_n$ be a **convergent** series of **positive terms**. Prove there exists a sequence of real numbers $\{c_n\}$ such that $\lim_{n\to\infty} c_n = \infty$ and such that $\sum_{n=1}^{\infty} c_n a_n$ converges.

3. Consider the series $\displaystyle\sum_{k=1}^{\infty} \frac{x}{k(1+kx^2)}$

   (a) Prove the series converges for all $x \in \mathbb{R}$.

   (b) Let $f(x) = \displaystyle\sum_{k=1}^{\infty} \frac{x}{k(1+kx^2)}$. Prove f is continuous at all $x \neq 0$.

   (c) Is f continuous at 0?

   Hint: Accept or prove (depending on how much time you have left)
$$\sum_{k=1}^{\infty} \frac{1}{k(1+kx^2)} \leq \frac{1}{1+x^2} + \int_1^{\infty} \frac{dt}{t(1+tx^2)}.$$
   But notice that the integral becomes infinite as $x \to 0$. As an additional hint $\dfrac{1}{t(1+t^2)} = \left(\dfrac{1}{t} - \dfrac{x^2}{1+x^2 t}\right)$. Do not separate prematurely!

4. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable; assume that $|f(x)| \leq 1$, $|f'(x)| \leq 1$ for all $x \in \mathbb{R}$ and that $f(0) = 0$. Let $\{a_n\}$ be a sequence of non zero real numbers. Prove: The sequence of functions
$$g_n(x) = \frac{1}{a_n} f(a_n x)$$
   has a subsequence converging to a continuous function.

5. Assume $f : \mathbb{R} \to \mathbb{R}$ and assume that $\{x \in \mathbb{R} : f(x) \geq r\}$ is measurable for each **rational** number r. Prove that f is measurable.

6. Assume f is Lebesgue integrable over the interval $[0,1]$. Prove that
$$\lim_{n\to\infty} \frac{1}{n} \int_0^1 \log\left(1 + e^{nf(x)}\right) dx = \int_{\{x\in[0,1]:f(x)>0\}} f(x)\, dx.$$
   Hint: Prove that $\log(1+e^t) \leq \log 2 + \max(t,0)$ for all $t \in \mathbb{R}$.

7. Let K be a compact subset of $\mathbb{R}$ with Lebesgue measure $m(K) = 1$. Let
$$K_0 = \bigcap \{A : A \text{ is a compact subset of } K \text{ and } m(A) = 1\}.$$
   Prove $m(K_0) = 1$ and if A is any proper compact subset of K_0, then $m(A) < 1$.

   Hint: Prove: The intersection of two, hence of any finite number, of compact subsets of K of measure 1, is again a compact subset of measure 1. Use this to conclude that if V is open in $\mathbb{R}$ and $K_0 \subset V$ then V must contain some compact subset of K of measure 1. Then use regularity of the Lebesgue measure; that is, use the fact that the measure of any measurable set is the infimum of the measure of all open sets containing it.

Analysis Qualifying Examination Spring 2010

Complete as many problems as possible.

1. Construct a subset of $[0,1]$ which is compact, perfect, nowhere dense, and with positive Lebesgue measure. (Be sure to prove your set has these four properties.)

2. [i] Suppose A, B are nonempty, disjoint, closed subsets of a metric space X. Show that the function $f : X \to [0,1]$ defined by
$$f(x) = \frac{\operatorname{dist}(x,A)}{\operatorname{dist}(x,A) + \operatorname{dist}(x,B)}$$
is continuous with $f(x) = 0$ for all $x \in A$, $f(x) = 1$ for all $x \in B$, and $0 < f(x) < 1$ for all $x \in X \setminus (A \cup B)$.

   [ii] Show that if X is a connected metric space with at least two distinct points, then X is uncountable.

3. [i] Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers. Prove the Root Test: if $\limsup_{n\to\infty} \sqrt[n]{|a_n|} < 1$, then $\sum a_n$ is absolutely convergent, and if $\limsup_{n\to\infty} \sqrt[n]{|a_n|} > 1$, then $\sum a_n$ is divergent.

   [ii] Let $c_n =$ (the nth digit of the decimal expansion of π) $+ 1$. Prove that the series $\sum c_n x^n$ has radius of convergence equal to 1.

4. Let $\{f_n\}$ be a sequence of continuously differentiable functions on $[0,1]$ with $f_n(0) = f_n'(0)$ and $|f_n'(x)| \leq 1$ for all $x \in [0,1]$ and $n \in \mathbb{N}$. Show that if $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in [0,1]$, then f is continuous on $[0,1]$. Must the sequence converge? Must there be a convergent subsequence?

5. Suppose that $f, g, h : [a,b] \to \mathbb{R}$ satisfy $f(x) \leq g(x) \leq h(x)$ for all $x \in [a,b]$ and $f(x_0) = h(x_0)$ for some $x_0 \in (a,b)$. Prove that if f and h are differentiable at x_0, then g is differentiable at x_0 with $f'(x_0) = g'(x_0) = h'(x_0)$.

6. Compute
$$\lim_{n\to\infty} \int_1^n \frac{n(\sqrt[n]{x} - 1)}{x^3 \log(x)}\, dx$$

7. Suppose $f : [a,b] \to \mathbb{R}$ is Riemann integrable on every subinterval $[a+\epsilon, b]$ for $0 < \epsilon < b - a$. Show that if f is Lebesgue integrable on $[a,b]$, then the (improper) Riemann integral exists on $[a,b]$ and is equal to the Lebesgue integral. Is the converse true?

FINAL DRAFT: Analysis Qualifying Examination Fall 2009

Complete as many problems as possible.

1. Let $u : [a, b+1] \to \mathbb{R}$ be a continuous function. For fixed $\tau \in [0,1]$ define $v_\tau : [a,b] \to \mathbb{R}$ by $v_\tau(t) = u(t+\tau)$. Show that $\{v_\tau \mid \tau \in [0,1]\}$ is a compact subset of $C([a,b],\mathbb{R})$.

2. Suppose A is a compact subset of $\mathbb{R}^n$ and $f : A \to \mathbb{R}$ is continuous. Prove that for every $\epsilon > 0$ there exists $M > 0$ such that $|f(x) - f(y)| \le M\|x - y\| + \epsilon$.

3. Suppose U is an open subset of $\mathbb{R}^n$ and $f : U \to \mathbb{R}^n$ is a homeomorphism. Prove that if f is uniformly continuous, then $U = \mathbb{R}^n$.

4. Suppose $\{C_n\}_{n\ge 1}$ is a nested decreasing sequence of nonempty, compact, connected subsets of a metric space. Prove that $\bigcap_{n\ge 1} C_n$ is nonempty, compact, and connected.

5. For $p > 1$ compute

$$\lim_{n\to\infty} \int_0^1 \frac{x^p}{x^2 + (1-nx)^2}\, dx.$$

6. Let $f : \mathbb{R} \to \mathbb{R}$ be a Lebesgue integrable function, $E \subset \mathbb{R}$ a measurable set, and $\omega > 0$. Define $\omega E = \{\omega x \mid x \in E\}$.

   [a] Show that $m(\omega E) = \omega\, m(E)$.

   [b] Show that the function $g : E \to \mathbb{R}$ defined by $g(x) = f(\omega x)$ is Lebesgue integrable and

$$\int_E f(\omega x)\, dx = \frac{1}{\omega} \int_{\omega E} f(x)\, dx.$$

7. Prove that if $f : [0,\infty) \to \mathbb{R}$ is Lebesgue integrable, then

$$\lim_{n\to\infty} \frac{1}{n}\, m\left(\left\{x \ge n \;\middle|\; |f(x)| \ge \frac{1}{n}\right\}\right) = 0.$$

Analysis Qualifying Examination

January 5, 2009

Name (Please print)____

1. Assume that $\{a_n\}$ is a monotone decreasing sequence with $a_n \geq 0$. If $\sum_{n=1}^{\infty} a_n < \infty$, show that $\lim_{n\to\infty} na_n = 0$. Is the converse true?

2. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x & \text{if } x \text{ is irrational} \\ p \sin \dfrac{1}{q} & \text{if } x = \dfrac{p}{q} \text{ in lowest terms } (q > 0) \end{cases}$$

   At what points is f continuous?

3. Let $f(x) = (x^2 - 1)^n$, and $g = f^{(n)}$ (i.e., the nth derivative of f.) Show that the polynomial g has n distinct real roots, all in the interval $[-1, 1]$.

4. Let X be a nonempty set, and for any two functions $f, g \in \mathbb{R}^X$ (the set of all functions from X to $\mathbb{R}$) let

$$d(f, g) = \sup_{x\in X} \frac{|f(x) - g(x)|}{1 + |f(x) - g(x)|}.$$

   Establish the following:

   (a) $(\mathbb{R}^X, d)$ is a metric space.

   (b) A sequence $\{f_n\} \subseteq \mathbb{R}^X$ satisfies $d(f_n, f) \to 0$ for some $f \in \mathbb{R}^X$ if and only if $\{f_n\}$ converges uniformly to f.

5. Let $E = \{(x, y) \in \mathbb{R}^2 : \ 9x^2 + y^4 = 1\}$. Show that E is compact and connected.

6. If $\int_A f = 0$ for every measurable subset A of a measurable set E, show that $f = 0$ a.e. in E.

7. Evaluate

$$\lim_{n\to\infty} \int_{[0,1]} \left(1 - e^{-x^2/n}\right) x^{-1/2} dx.$$

Analysis Qualifying Examination

August 21, 2008

Name (Please print)____

1. If $\{x_n\}$ is a convergent sequence in a metric space, show that any rearrangement of $\{x_n\}$ converges to the same limit.

2. Consider the series
$$\sum_{n=1}^{\infty} \frac{1}{1+x^n}.$$
   (a) Show that the series diverges for $|x| < 1$, and converges for $|x| > 1$.
   (b) Let $f(x) = \sum_{n=1}^{\infty} \frac{1}{1+x^n}$. Find the set where f is continuous.

3. Let G be a non-trivial additive subgroup of $\mathbb{R}$. Let
$$a = \inf \{x \in G : x > 0\}.$$
Prove: If $a > 0$ then $G = \{na : n \in \mathbb{Z}\}$, otherwise (i.e., if $a = 0$) G is dense in $\mathbb{R}$.

4. Consider the function $f : [-1, 1] \longrightarrow \mathbb{R}$ defined by
$$f(x) = \begin{cases} \dfrac{x}{2} + x^2 \sin\left(\dfrac{1}{x}\right), & \text{if } x \neq 0; \\ 0, & \text{if } x = 0. \end{cases}$$
Prove that $f'(0)$ is positive, but that f is not increasing in any open interval that contains 0.

5. For $x \in [-1, 1]$ and $n \in \mathbb{N}$, define
$$f_n(x) = \frac{x^{2n}}{1+x^{2n}}.$$
   (a) Find a function f_0 on $[-1, 1]$ such that $\{f_n\}$ converges pointwise to f_0.
   (b) Determine whether $\{f_n\}$ converges uniformly to f_0.
   (c) Calculate $\int_{-1}^{1} f_0(x)dx$ and determine whether
$$\lim_{n\to\infty} \int_{-1}^{1} f_n(x)dx = \int_{-1}^{1} f_0(x)dx.$$

6. Let $(X, \mathcal{A}, \mu)$ be a measurable space and $\{f_n\}$ a sequence of measurable functions. We say that $\{f_n\}$ converges in measure to f if for every $\varepsilon > 0$,
$$\lim_{n\to\infty} \mu\left(\{x \in X : |f_n(x) - f(x)| > \varepsilon\}\right) = 0.$$
Show that if μ is a finite measure, and $f_n \longrightarrow f$ a.e., then $f_n \longrightarrow f$ in measure. Give an example of a sequence which converges in measure, but does not converge a.e.

7. Show that
$$\lim_{n\to\infty} \int_0^n \left(1 + \frac{x}{n}\right)^n e^{-2x} dx = 1.$$

Analysis Qualifying Examination Spring 2008

Complete as many problems as possible.

1. Let $f(x) = \sin\left(\frac{1}{x}\right)$ for $x > 0$. Prove that f is uniformly continuous on (a, ∞) for every $a > 0$. Is f uniformly continuous on $(0, \infty)$? Justify your answer.

2. Let f be continuous on $[0, 1]$ and differentiable in $(0, 1)$ such that $f(0) = f(1)$. Prove that there exists $0 < c < 1$ such that $f'(1 - c) = -f'(c)$.

3. Suppose X, Y are metric spaces, X is compact, and $f : X \to Y$ is a continuous bijection. Prove that f is a homeomorphism.

4. Let $C([0, 1], \mathbb{R}) = \{f \mid [0, 1] \to \mathbb{R} \mid f \text{ is continuous}\}$ be the metric space of continuous functions on $[0, 1]$ with the metric $d(f, g) = \|f - g\|_\infty = \sup_{x\in[0,1]} |f(x) - g(x)|$. Show that the unit ball $\{f \in C([0, 1], \mathbb{R}) \mid \|f\|_\infty \leq 1\}$ is not compact.

5. Let $f : [0, 1] \to \mathbb{R}$ be continuously differentiable on $[0, 1]$ with $f(0) = 0$. Prove that

$$\sup_{0\leq x\leq 1} |f(x)| \leq \int_0^1 |f'(x)|\, dx \leq \left(\int_0^1 |f'(x)|^2\, dx\right)^{\frac{1}{2}}$$

6. Compute

$$\lim_{n\to\infty} \int_0^1 \sqrt[3]{1 + x^n \sin(nx)}\, dx.$$

   Justify your answer.

7. Let $f : \mathbb{R} \to \mathbb{R}$ be Lebesgue integrable on $\mathbb{R}$. Prove that

$$\lim_{h\to 0} \int_{\mathbb{R}} |f(x + h) - f(x)|\, dx = 0.$$

Analysis Qualifying Examination Fall 2007

Name: ________________ **Last Four Digits of Your Student Number:** ________________

In some cases partial credit may be given, but you should endeavor to fully complete as many problems as possible.

1. Give an example of a sequence of Riemann integrable functions $f_n : [0,1] \to \mathbb{R}$ for which $f_n \to 0$ pointwise on [0,1] but
$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx \neq 0.$$

2. Let k be a fixed positive integer, and let A be the set of all polynomials of the form
$$p(x) = a_k x^k + a_{k+1}x^{k+1} + \cdots + a_n x^n, \text{ where } n \in \mathbb{N}, n \geq k, \text{ and } a_i \in \mathbb{R}.$$
Prove that A is dense in $C[a,1]$ for every $0 < a < 1$. Is it also dense in $C[0,1]$? Prove your conclusion.

3. (a) Suppose that $f : [a,b] \to \mathbb{R}$ is differentiable, and that f' is bounded on $[a,b]$. Prove that f is of bounded variation on $[a,b]$. (b) Define $f(x) = x^2\cos(1/x)$ for $0 < x \leq 1$ and $f(0) = 0$. Using (a), prove that f is of bounded variation on $[0,1]$.

4. Let U be an open subset of $\mathbb{R}^n$ and let $f : U \to \mathbb{R}^m$. (a) State the definition that f is differentiable at $p \in U$. (b) Show that every linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at every $p \in \mathbb{R}^n$. What is the derivative of T at p?

5. Define $f : [0,1] \to \mathbb{R}$ by
$$f(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ x^2, & \text{if } x \text{ is irrational} \end{cases}$$
Prove that f is not Riemann integrable on $[0,1]$ but it is Lebesgue integrable on $[0,1]$. Find the Lebesgue integral of f.

6. Let X be a compact metric space and let $\{f_n\}$ be a sequence of isometries on X. Prove that there exists a subsequence $\{f_{n_k}\}$ that pointwise converges to an isometry f on X. Recall that $f : X \to X$ is called an isometry if $d(f(x), f(y)) = d(x,y)$ for all $x, y \in X$, where d is the metric on X.

Analysis Qualifying Examination Spring 2007

Name: [redacted] Last Four Digits of Your Student Number: [redacted]

Full completion of at least 4 problems will result in a pass. In some cases partial credit may be given, but you should endeavor to fully complete as many problems as possible.

1. Define the sequence $\{a_n\}$ by setting
$$a_n = \frac{n!}{n^n}$$
for all positive integers n.
   (a) Prove that the sequence $\{a_n\}$ is strictly decreasing and bounded.
   (b) Find the limit of the sequence $\{a_n\}$.
   (c) Find the limit of the sequence $\left\{\frac{a_{n+1}}{a_n}\right\}$.

2. Let A and B be two nonempty open subsets of $\mathbb{R}$, the set of real numbers equipped with the usual metric. Suppose that $x < y$ for all $x \in A$ and all $y \in B$. Prove that there exists a real number z such that $x < z < y$ for all $x \in A$ and all $y \in B$.

3. Consider the series
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^x},$$
where x is a real variable. Find the values of x for which the series converges and find the values of x for which the series converges absolutely. Show that the function
$$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^x}$$
defined for $x > 0$ is continuous on the interval $(0, +\infty)$.

4. Suppose that $f : (0, +\infty) \to \mathbb{R}$ is differentiable and that there exists a constant $\alpha \in \mathbb{R}$ such that $xf'(x) = \alpha f(x)$ for all $x > 0$ and $f(1) = 1$. Prove that $f(x) = x^\alpha$ for all $x > 0$.

5. Let K be a compact subset of $\mathbb{R}^n$ and let $f : K \to \mathbb{R}$ be a continuous function. Define
$$M = \{x \in K : f(x) \text{ is the maximum value of } f \text{ on } K\}.$$
Prove that M is a compact subset of $\mathbb{R}^n$.

6. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a function such that $\|f(x) - f(y)\| \leq \|x - y\|^2$ for all $x, y \in \mathbb{R}^n$, where $\|\cdot\|$ denotes the Euclidean norm. Prove that f is a constant function.

Analysis Qualifying Examination Fall 2006

Name: ________________ Last Four Digits of Your Student Number: ________________ Fall 2006

Full completion of at least 4 problems will result in a pass. In some cases partial credit may be given, but you should endeavor to fully complete as many problems as possible.

1. Let A be a bounded subset of $\mathbb{R}$. Define $D = \{x - y : x \in A \text{ and } y \in A\}$. Prove that D is bounded and $\sup(D) = \sup(A) - \inf(A)$. State and prove a similar result for $\inf(D)$.

2. Prove that the series
$$\sum_{n=1}^{\infty} \left(\frac{x^n}{n} - \frac{x^{n+1}}{n+1} \right)$$
converges uniformly on the interval $[-1, 1]$.

3. Let $f : (0,1) \to \mathbb{R}$ be differentiable such that f' is bounded on $(0,1)$. Prove that both $\lim_{x \to 0+} f(x)$ and $\lim_{x \to 1-} f(x)$ exist.

4. Let $f(0,0) = 0$ and
$$f(x,y) = \frac{2xy(x^2 - y^2)}{x^2 + y^2}$$
if $(x,y) \neq (0,0)$. Prove that f is continuous everywhere and the two partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist everywhere. Is f differentiable at $(0,0)$? Justify your conclusion.

5. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous and satisfies the equation
$$f(x) = 1 + \int_0^x f(t)\, dt$$
for all real numbers x. Prove that $f(x) = e^x$.

6. Let $f : [a,b] \to \mathbb{R}$ be continuous, where $[a,b]$ is a bounded closed interval. Define
$$G = \{(x, f(x)) : x \in [a,b]\}.$$
G is called the graph of f. Prove that G is a connected subset of $\mathbb{R}^2$, where $\mathbb{R}^2$ is equipped with the standard metric.

Analysis Qualifying Examination — Spring 2006

Name: ________________ **Last Four Digits of Your Student Number:** ________________

Full completion of at least 4 problems will result in a pass. In some cases partial credit may be given, but you should endeavor to fully complete as many problems as possible.

1. (a) Show that $|\sin x - \sin y| \leq |x - y|$ for all real numbers x and y.

   (b) Show that there exists no positive constant $0 < c < 1$ such that $|\sin x - \sin y| \leq c|x - y|$ for all real numbers x and y.

2. Define a sequence $\{x_n\}$ as follows:

$$x_n = \frac{1}{1 \cdot 4} + \frac{1}{2 \cdot 5} + \frac{1}{3 \cdot 6} + \cdots + \frac{1}{n \cdot (n+3)}.$$

   Prove that $\{x_n\}$ converges and find its limit.

3. Assume that f is differentiable at a and that $\{x_n\}$ and $\{z_n\}$ are two sequences that converge to a and that $x_n < a < z_n$ for all n. Prove that

$$\lim_{n \to \infty} \frac{f(x_n) - f(z_n)}{x_n - z_n} = f'(a).$$

4. Assume that f is a continuous real-valued function on $\mathbb{R}$ and that

$$\lim_{x \to +\infty} f(x) = \lim_{x \to -\infty} f(x) = 0.$$

   Prove that if f does not have a maximum then it has a minimum and if it does not have a minimum then it has a maximum.

5. Prove that the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{x^2 + n}{n^2}$$

   converges uniformly in every bounded interval, but does not converge absolutely for any real value of x.

6. Suppose that (M, d) is a metric space and that $\{x_n\}$ is a Cauchy sequence in (M, d). Suppose also that $\{x_n\}$ has a convergent subsequence. Prove that $\{x_n\}$ converges.

Fall 2005

1. For every positive integer n, define

$$a_n = \int_0^1 \frac{x^n}{1+x^n}\,dx.$$

Prove that $0 \le a_{n+1} \le a_n$ for all n. Is the sequence $\{a_n\}$ convergent? If so, find its limit.

2. Let $f : I \to \mathbb{R}$ be continuous, where $I = (a, b)$ is an open interval in $\mathbb{R}$. Assume that $c \in I$ and f is differentiable in (a, c) and in (c, b), and assume that $L = \lim_{x\to c} f'(x)$ exists (and is finite). Prove that f is differentiable at c and $f'(c) = L$.

3. Consider the series

$$\sum_{n=0}^{\infty}(e^{-nx} - e^{-2nx}).$$

(a) Prove that it converges for all $x \ge 0$.

(b) For $x \ge 0$, set

$$f(x) = \sum_{n=0}^{\infty}(e^{-nx} - e^{-2nx}).$$

Prove that f is continuous at all $x > 0$ but discontinuous at $x = 0$.

4. Let f and g be two real valued differentiable functions defined on $\mathbb{R}$. Suppose that for all $x \in \mathbb{R}$

$$f^2(x) + g^2(x) = 1$$
$$f'(x) = g(x) \text{ and } g'(x) = -f(x)$$
$$f(0) = 0 \text{ and } g(0) = 1.$$

Prove that f is the sine function and g is the cosine function.
Hint: Consider the function $F(x) = (f(x) - \sin x)^2 + (g(x) - \cos x)^2$.

5. Let C be a connected subset of $\mathbb{R}$. Prove that if $f : C \to \mathbb{R}$ is continuous and $f(x)$ is an integer for all $x \in C$, then f is a constant function.

6. Let U be an open subset of $\mathbb{R}^n$ and let $f : U \to U$ be one-to-one and onto and differentiable in U. Let f^{-1} denote the inverse of f. Prove that if $f'(c) = 0$ for some $c \in U$, then f^{-1} is not differentiable in U, where $f'(c)$ denotes the differential of f at c; another notation is $df(c)$.

Analysis Qualifying Examination Fall 2004

You should try all the six problems. But please tell us which five of them you want us to grade.

1. If f is continuous on $[a,b]$, differentiable in (a,b), and $f'(x) = 0$ for all $x \in (a,b)$, prove that f is a constant on $[a,b]$.

2. Let f be a function defined on $[0,+\infty)$ such that (i) f is continuous on $[0,+\infty)$, (ii) f is differentiable in $(0,+\infty)$, (iii) $f'(x) \geq 1$ for all $x > 0$, and (iv) $f(0) = 0$. Fix $x_0 > 0$ and define $x_n = f(x_{n-1})$ for all $n \geq 1$.

   (a) Prove that $x_{n+1} \geq x_n$ for all n.

   (b) Prove that if the sequence $\{x_n\}$ converges then $\{x_n\}$ is a constant sequence, that is, $x_n = x_1$ for all $n \geq 1$.

3. Given the power series
$$\sum_{n=1}^{\infty} \frac{2^n n!}{1 \cdot 3 \cdot 5 \cdot 7 \cdots (2n-1)} x^n.$$
Find all x for which the series converges.

4. Let f be a decreasing function defined on $[1,+\infty)$. Suppose that $f(x) > 0$ for all $x \geq 1$. Prove that the sequence $\{D_n\}$ defined by
$$D_n = f(1) + f(2) + \cdots + f(n) - \int_1^n f(t)\,dt$$
is non-negative and decreasing and hence converges.

5. Let $\mathbf{R}$ be the set of real numbers equipped with the usual metric. Prove that every uncountable subset of $\mathbf{R}$ has at least one limit point.

6. Let (X,d) be a compact metric space. Let $f : X \to X$ be a function such that $d(f(x), f(y)) < d(x,y)$ for all $x, y \in X$ with $x \neq y$. (a) Prove that the function F defined by $F(x) = d(f(x), x)$ for $x \in X$ is continuous on X. (b) Using (a) prove that f has a unique fixed point, that is, there exists a unique $z \in X$ such that $f(z) = z$.

Analysis Qualifying Examination Spring 2005

Solve as many problems as you can. You do not have to solve all to pass the qualifying exam.

1. Let f be defined for all real x, and suppose that

$$|f(x) - f(y)| \leq (x - y)^2$$

for all real x and y. Prove that f is differentiable and $f'(x) = 0$ for all x. What can you say about the function f?

2. If $0 < x < \frac{\pi}{2}$, prove that

$$\frac{2}{\pi} < \frac{\sin x}{x} < 1.$$

3. (a) Let f be a continuous function on $[a, b]$ with $a < b$. Suppose that $f(x) \geq 0$ for all $x \in [a, b]$ and $\int_a^b f(x)\,dx = 0$. Prove that $f(x) = 0$ for all $x \in [a, b]$.

(b) Let g be a real continuous function on $[0, 1]$ such that

$$\int_0^1 g^2(x)\,dx = \frac{1}{3} \quad \text{and} \quad \int_0^1 xg(x)\,dx = \frac{1}{3}.$$

Using (a), prove that $g(x) = x$ for all $x \in [0, 1]$.

4. Given the power series

$$\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}.$$

Find all real x for which the series converges.

5. Let X be a nonempty set. For all $x, y \in X$, define $d(x, y) = 1$ if $x \neq y$ and $d(x, y) = 0$ if $x = y$. Prove that d is a metric on X. Which subsets of the resulting metric space (X, d) are open? Which are closed? Which are compact? Prove your statements.

6. Define $f(0, 0) = 0$ and

$$f(x, y) = \frac{x^3}{x^2 + y^2} \quad \text{if } (x, y) \neq (0, 0).$$

(a) Prove that f is continuous at $(0, 0)$.

(b) Prove that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist for all x and y.